## Section 11.3: Length Contraction, Simultaneity, and Relativistic Momentum

### Tutorial 1 Practice, page 591

**1. Given:** $L_s$ = 5.0 m; $L_m$ = 4.5 m; $c = 3.0\times10^8$ m/s
**Required:** $v$

**Analysis:** $$\frac{L_m}{L_s} = \sqrt{1-\frac{v^2}{c^2}}$$

$$\left(\frac{L_m}{L_s}\right)^2 = 1-\frac{v^2}{c^2}$$

$$\frac{v^2}{c^2} = 1-\left(\frac{L_m}{L_s}\right)^2$$

$$v = c\sqrt{1-\left(\frac{L_m}{L_s}\right)^2}$$

**Solution:** $$v = c\sqrt{1-\left(\frac{L_m}{L_s}\right)^2}$$

$$= (3.0\times10^8 \text{ m/s})\sqrt{1-\frac{(4.5 \cancel{\text{m}})^2}{(5.0 \cancel{\text{m}})^2}}$$

$$v = 1.3\times10^8 \text{ m/s}$$

**Statement:** To have had a relativistic contraction to 4.5 m, the 5.0 m long object must have moved at $1.3 \times 10^8$ m/s.

**2. Given:** $L_s$ = 120 m; $v = 0.80c$
**Required:** $L_m$

**Analysis:** $$\frac{L_m}{L_s} = \sqrt{1-\frac{v^2}{c^2}}$$

$$L_m = L_s\sqrt{1-\frac{v^2}{c^2}}$$

**Solution:** $$L_m = L_s\sqrt{1-\frac{v^2}{c^2}}$$

$$= (120 \text{ m})\sqrt{1-\frac{(0.80\cancel{c})^2}{\cancel{c^2}}}$$

$$L_m = 72 \text{ m}$$

**Statement:** The relativistic contraction reduces the length of the spacecraft to 72 m.

**3. (a) Given:** $L_s$ = 2.5 m; $L_m$ = 2.2 m; $c = 3.0 \times 10^8$ m/s
**Required:** $v$

**Analysis:** $$\frac{L_m}{L_s} = \sqrt{1-\frac{v^2}{c^2}}$$

$$\left(\frac{L_m}{L_s}\right)^2 = 1-\frac{v^2}{c^2}$$

$$\frac{v^2}{c^2} = 1-\left(\frac{L_m}{L_s}\right)^2$$

$$v = c\sqrt{1-\left(\frac{L_m}{L_s}\right)^2}$$

**Solution:** $$v = c\sqrt{1-\left(\frac{L_m}{L_s}\right)^2}$$

$$= (3.0\times10^8 \text{ m/s})\sqrt{1-\frac{(2.2 \cancel{\text{m}})^2}{(2.5 \cancel{\text{m}})^2}}$$

$$v = 1.4\times10^8 \text{ m/s}$$

**Statement:** To have contracted from 2.5 m to 2.2 m, the car must have moved at $1.4 \times 10^8$ m/s.

**(b) Given:** $L_s$ = 33 m; $L_m$ = 26 m; $c = 3.0 \times 10^8$ m/s
**Required:** $v$

**Analysis:** Same as in part (a) above, $v = c\sqrt{1-\left(\frac{L_m}{L_s}\right)^2}$.

**Solution:** $$v = c\sqrt{1-\left(\frac{L_m}{L_s}\right)^2}$$

$$= (3.0\times10^8 \text{ m/s})\sqrt{1-\frac{(26 \cancel{\text{m}})^2}{(33 \cancel{\text{m}})^2}}$$

$$v = 1.8\times10^8 \text{ m/s}$$

**Statement:** To have contracted from 33 m to 26 m, the rocket must have moved at $1.8 \times 10^8$ m/s.

## Tutorial 2 Practice, page 596

**1. (a) Given:** $m = 1.67 \times 10^{-27}$ kg; $v = 0.85c$; $c = 3.0 \times 10^8$ m/s

**Required:** $p_{\text{classical}}$

**Analysis:** $p_{\text{classical}} = mv$

**Solution:**
$$\begin{aligned} p_{\text{classical}} &= mv \\ &= m\,(0.85c) \\ &= (1.67 \times 10^{-27}\ \text{kg})(0.85)(3.0 \times 10^8\ \text{m/s}) \\ &= 4.259 \times 10^{-19}\ \text{kg} \cdot \text{m/s (two extra digits carried)} \end{aligned}$$
$$p_{\text{classical}} = 4.3 \times 10^{-19}\ \text{kg} \cdot \text{m/s}$$

**Statement:** The proton's classical momentum is $4.3 \times 10^{-19}\ \text{kg} \cdot \text{m/s}$.

**(b) Given:** $m = 1.67 \times 10^{-27}$ kg; $v = 0.85c$; $p_{\text{classical}} = 4.259 \times 10^{-19}\ \text{kg} \cdot \text{m/s}$

**Required:** $p_{\text{relativistic}}$

**Analysis:** $p_{\text{relativistic}} = \dfrac{mv}{\sqrt{1 - \dfrac{v^2}{c^2}}}$ and $p_{\text{classical}} = mv$, so $p_{\text{relativistic}} = \dfrac{p_{\text{classical}}}{\sqrt{1 - \dfrac{v^2}{c^2}}}$.

**Solution:**
$$\begin{aligned} p_{\text{relativistic}} &= \frac{p_{\text{classical}}}{\sqrt{1 - \dfrac{v^2}{c^2}}} \\ &= \frac{4.259 \times 10^{-19}\ \text{kg} \cdot \text{m/s}}{\sqrt{1 - \dfrac{(0.85\cancel{c})^2}{\cancel{c^2}}}} \end{aligned}$$
$$p_{\text{relativistic}} = 8.1 \times 10^{-19}\ \text{kg} \cdot \text{m/s}$$

**Statement:** The proton's relativistic momentum in the lab frame of reference is $8.1 \times 10^{-19}\ \text{kg} \cdot \text{m/s}$, about twice the classical value.

**2. Given:** $m = 0.1$ kg; $v = 0.30c$; $c = 3.0 \times 10^8$ m/s

**Required:** $p_{\text{relativistic}}$

**Analysis:** $p_{\text{relativistic}} = \dfrac{mv}{\sqrt{1 - \dfrac{v^2}{c^2}}}$

**Solution:** $p_{\text{relativistic}} = \dfrac{mv}{\sqrt{1-\dfrac{v^2}{c^2}}}$

$= \dfrac{m(0.30c)}{\sqrt{1-\dfrac{(0.30\cancel{c})^2}{\cancel{c^2}}}}$

$= \dfrac{(0.1\text{ kg})(0.30)(3.0\times10^8\text{ m/s})}{\sqrt{0.91}}$

$p_{\text{relativistic}} = 9.4\times10^6\text{ kg}\cdot\text{m/s}$

**Statement:** The projectile's relativistic momentum with respect to Earth is $9.4\times10^6\text{ kg}\cdot\text{m/s}$.

**3. Given:** $m = 1.67\times10^{-27}$ kg; $v = 0.750c$; $c = 3.0\times10^8$ m/s

**Required:** $p_{\text{relativistic}}$

**Analysis:** $p_{\text{relativistic}} = \dfrac{mv}{\sqrt{1-\dfrac{v^2}{c^2}}}$

**Solution:** $p_{\text{relativistic}} = \dfrac{mv}{\sqrt{1-\dfrac{v^2}{c^2}}}$

$= \dfrac{m(0.750c)}{\sqrt{1-\dfrac{(0.750\cancel{c})^2}{\cancel{c^2}}}}$

$= \dfrac{(1.67\times10^{-27}\text{ kg})(0.750)(3.0\times10^8\text{ m/s})}{\sqrt{0.4375}}$

$p_{\text{relativistic}} = 5.68\times10^{-19}\text{ kg}\cdot\text{m/s}$

**Statement:** The proton's relativistic momentum in the lab frame of reference is $5.68\times10^{-19}\text{ kg}\cdot\text{m/s}$.

**4. (a)** The motion affects only the component of length along the direction of motion. So, only direction $y$ is affected.

**(b) Given:** $L_{\text{xs}} = 0.100$ m; $L_{\text{ys}} = 0.100$ m; $L_{\text{zs}} = 0.100$ m (proper lengths of the cube); $v_y = 0.950c$ (speed along $y$-axis); $c = 3.0\times10^8$ m/s

**Required:** relativistic volume of the cube, $V_{\text{m}}$

**Analysis:** $V_{\text{m}} = L_{x\text{m}}L_{y\text{m}}L_{z\text{m}}$. We know that $x$- and $z$-directions are unaffected by the motion, so $L_{x\text{m}} = L_{xs}$ and $L_{z\text{m}} = L_{zs}$.

Use $\frac{L_\text{m}}{L_\text{s}} = \sqrt{1-\frac{v^2}{c^2}}$, rearranged to solve for $L_\text{m}$.

We only need to calculate $L_{y\text{m}} = L_{y\text{s}}\sqrt{1-\frac{v^2}{c^2}}$.

**Solution:** $V_\text{m} = L_{x\text{m}}L_{y\text{m}}L_{z\text{m}}$

$$= L_{x\text{s}}L_{y\text{s}}L_{z\text{s}}\sqrt{1-\frac{v^2}{c^2}}$$

$$= (0.100\text{ m})(0.100\text{ m})(0.100\text{ m})\sqrt{1-\frac{(0.950\cancel{c})^2}{\cancel{c^2}}}$$

$$V_\text{m} = 3.12\times10^{-4}\text{ m}^3$$

$$V_\text{m} = L_\text{xs}L_\text{ys}L_\text{zs}\ \sqrt{1-(v/c)^2}$$

$$= (0.100\text{ m})(0.100\text{ m})(0.100\text{ m})\ \sqrt{1-(0.950)^2}$$

$$V_\text{m} = 3.12\times10^{-4}\text{ m}^3$$

**Statement:** The cube contracts along its direction of motion, resulting in a relativistic volume of $3.12\times10^{-4}\text{ m}^3$. This is less than the proper volume $V_\text{s} = 1.00\times10^{-3}\text{ m}^3$ $(= L_{x\text{s}}L_{y\text{s}}L_{z\text{s}})$.

**(c) Given:** $V_\text{s} = 1.00\times10^{-3}\text{ m}^3$; *density* $= 2.26\times10^4\text{ kg/m}^3$; $v = 0.950c$; $c = 3.0\times10^8\text{ m/s}$

**Required:** $p_\text{relativistic}$

**Analysis:** First, use the proper volume and density to calculate the rest mass, *m*. Then, use $p_\text{relativistic} = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}$.

**Solution:** $p_\text{relativistic} = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}$

$$= \frac{(V_\text{s}\cdot\text{density})(0.950c)}{\sqrt{1-\frac{(0.950\cancel{c})^2}{\cancel{c^2}}}}$$

$$= \frac{(1.00\times10^{-3}\text{ m}^3)(2.26\times10^4\text{ kg/m}^3)(0.950)(3.0\times10^8\text{ m/s})}{0.3122}$$

$$p_\text{relativistic} = 2.06\times10^{10}\text{ kg}\cdot\text{m/s}$$

**Statement:** The cube's relativistic momentum is $2.06\times10^{10}\text{ kg}\cdot\text{m/s}$.

## Section 11.3 Questions, page 597

**1. Given:** $L_{\mathrm{m}} = 475\text{ m};\ v = 0.755c$

**Required:** $L_{\mathrm{s}}$

**Analysis:** $\dfrac{L_{\mathrm{m}}}{L_{\mathrm{s}}} = \sqrt{1-\dfrac{v^2}{c^2}}$, so $L_{\mathrm{s}} = \dfrac{L_{\mathrm{m}}}{\sqrt{1-\dfrac{v^2}{c^2}}}$.

**Solution:**
$$\begin{aligned} L_{\mathrm{s}} &= \frac{L_{\mathrm{m}}}{\sqrt{1-\dfrac{v^2}{c^2}}} \\ &= \frac{475\text{ m}}{\sqrt{1-\dfrac{(0.755\cancel{c})^2}{\cancel{c^2}}}} \\ L_{\mathrm{s}} &= 724\text{ m} \end{aligned}$$

**Statement:** The proper length of spacecraft 2 is 724 m.

**2. Given:** $L_{\mathrm{m1}} = 8.0\text{ ly};\ v_1 = 0.55c;\ v_2 = 0.85c$

**Required:** $L_{\mathrm{m2}}$

**Analysis:** Apply the length contraction formula to each astronaut (the proper lengths are the same).

$$\frac{L_{\mathrm{m1}}}{L_{\mathrm{s}}} = \sqrt{1-\frac{v_1^2}{c^2}}\ ;\ \frac{L_{\mathrm{m2}}}{L_{\mathrm{s}}} = \sqrt{1-\frac{v_2^2}{c^2}}$$

Divide the relation for astronaut 2 by that for astronaut 1.

$$\frac{L_{\mathrm{m2}}}{L_{\mathrm{m1}}} = \frac{\sqrt{1-\dfrac{v_2^2}{c^2}}}{\sqrt{1-\dfrac{v_1^2}{c^2}}},\ \text{so}\ L_{\mathrm{m2}} = L_{\mathrm{m1}}\frac{\sqrt{1-\dfrac{v_2^2}{c^2}}}{\sqrt{1-\dfrac{v_1^2}{c^2}}}$$

**Solution:** $L_{m2} = L_{m1}\dfrac{\sqrt{1-\dfrac{v_2^2}{c^2}}}{\sqrt{1-\dfrac{v_1^2}{c^2}}}$

$$= (8.0 \text{ ly})\frac{\sqrt{1-\dfrac{(0.85\cancel{c})^2}{\cancel{c^2}}}}{\sqrt{1-\dfrac{(0.55\cancel{c})^2}{\cancel{c^2}}}}$$

$$L_{m2} = 5.0 \text{ ly}$$

**Statement:** The second astronaut, who travels at 0.85*c*, finds the distance to the star to be 5.0 ly.

**3.** François does not observe the explosions to occur simultaneously. Assume François is travelling east relative to Soledad. According to him, Soledad is moving west at a speed of 0.95*c*. Now Soledad has arranged the explosions in her moving frame to occur such that the light from each one reaches her at the same time in her inertial frame of reference. However, once the light leaves the explosion sites, they travel at speed *c*. So, if François had seen the explosions as simultaneous, then Soledad would have detected the western explosion first, because she was moving toward that explosion. Since she instead observed the explosions simultaneously, the western explosion must have occurred later than the eastern explosion. In other words, François sees the explosion next to the front of his railway car first.

**4. (a) Given:** $m = 1.67\times10^{-27}$ kg; $v = 0.99c$; $c = 3.0\times10^{8}$ m/s

**Required:** $p_{\text{classical}}$

**Analysis:** $p_{\text{classical}} = mv$

**Solution:** $p_{\text{classical}} = mv$

$$= m(0.99c)$$

$$= (1.67\times10^{-27}\text{ kg})(0.99)(3.0\times10^{8}\text{ m/s})$$

$$= 4.960\times10^{-19}\text{ kg}\cdot\text{m/s (two extra digits carried)}$$

$$p_{\text{classical}} = 5.0\times10^{-19}\text{ kg}\cdot\text{m/s}$$

**Statement:** This proton’s momentum, according to Newton’s definition, is $5.0\times10^{-19}\text{ kg}\cdot\text{m/s}$.

**(b) Given:** $p_{\text{classical}} = 4.960\times10^{-19}\text{ kg}\cdot\text{m/s}$; $v = 0.99c$

**Required:** $p_{\text{relativistic}}$

**Analysis:** $p_{\text{relativistic}} = \dfrac{mv}{\sqrt{1-\dfrac{v^2}{c^2}}}$ and $p_{\text{classical}} = mv$, so $p_{\text{relativistic}} = \dfrac{p_{\text{classical}}}{\sqrt{1-\dfrac{v^2}{c^2}}}$. Calculate the relativistic factor separately because it will be used in part (c) below.

**Solution:** $p_{\text{relativistic}} = \dfrac{p_{\text{classical}}}{\sqrt{1-\dfrac{v^2}{c^2}}}$

$$= \frac{4.960\times10^{-19}\ \text{kg}\cdot\text{m/s}}{\sqrt{1-\dfrac{(0.99\cancel{c})^2}{\cancel{c^2}}}}$$

$$= \frac{4.960\times10^{-19}\ \text{kg}\cdot\text{m/s}}{7.088}\quad\text{(two extra digits carried)}$$

$$p_{\text{relativistic}} = 3.52\times10^{-18}\ \text{kg}\cdot\text{m/s}$$

**Statement:** The proton's relativistic momentum is $3.52\times10^{-18}\ \text{kg}\cdot\text{m/s}$.

**(c)** From the intermediate step in part (b) above, the relativistic momentum exceeds that of the classical value by a ratio of 7 : 1.

**5. Given:** $p_{\text{relativistic}} / p_{\text{classical}} = 5$

**Required:** $v$

**Analysis:** $p_{\text{relativistic}} = \dfrac{mv}{\sqrt{1-\dfrac{v^2}{c^2}}}$ and $p_{\text{classical}} = mv$, so $p_{\text{relativistic}} = \dfrac{p_{\text{classical}}}{\sqrt{1-\dfrac{v^2}{c^2}}}$, or

$\dfrac{p_{\text{relativistic}}}{p_{\text{classical}}} = \dfrac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}$. Rewrite this equation in terms of $\dfrac{v}{c}$.

$$\frac{p_{\text{relativistic}}}{p_{\text{classical}}} = \frac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}$$

$$\left(\frac{p_{\text{relativistic}}}{p_{\text{classical}}}\right)^2 = \frac{1}{1-\dfrac{v^2}{c^2}}$$

$$1-\frac{v^2}{c^2} = \frac{1}{\left(\dfrac{p_{\text{relativistic}}}{p_{\text{classical}}}\right)^2}$$

$$\frac{v^2}{c^2} = 1-\frac{1}{\left(\dfrac{p_{\text{relativistic}}}{p_{\text{classical}}}\right)^2}$$

$$\frac{v}{c} = \sqrt{1-\frac{1}{\left(\dfrac{p_{\text{relativistic}}}{p_{\text{classical}}}\right)^2}}$$

**Solution:** $\dfrac{v}{c} = \sqrt{1 - \dfrac{1}{\left(\dfrac{p_{\text{relativistic}}}{p_{\text{classical}}}\right)^2}}$

$$= \sqrt{1 - \frac{1}{25}}$$

$$\frac{v}{c} = 0.980$$

$$v = 0.980c$$

**Statement:** To have increased its momentum by a relativistic factor of 5, the speed of the particle must be 0.98$c$.

**6. Given:** $m_{\text{electron}} = 9.11 \times 10^{-31}\ \text{kg}$; $v_{\text{electron}} = 0.999c$; $m_{\text{ship}} = 4.38 \times 10^{7}\ \text{kg}$; $c = 3.0 \times 10^{8}\ \text{m/s}$

**Required:** $v_{\text{ship}}$

**Analysis:** The speed of the ship will be much less than that of the electron, so its relativistic momentum will be very close to its classical momentum. This is not the case for the electron, so we will use the relativistic expression for just the electron.

$p_{\text{ship}} = m_{\text{ship}} v_{\text{ship}}$, $p_{\text{electron}} = \dfrac{m_{\text{electron}} v_{\text{electron}}}{\sqrt{1 - \dfrac{(v_{\text{electron}})^2}{c^2}}}$, and $p_{\text{ship}} = p_{\text{electron}}$, so

$$m_{\text{ship}} v_{\text{ship}} = \frac{m_{\text{electron}} v_{\text{electron}}}{\sqrt{1 - \dfrac{(v_{\text{electron}})^2}{c^2}}}$$

$$v_{\text{ship}} = \frac{m_{\text{electron}} v_{\text{electron}}}{m_{\text{ship}} \sqrt{1 - \dfrac{(v_{\text{electron}})^2}{c^2}}}$$

**Solution:** $v_{\text{ship}} = \dfrac{m_{\text{electron}} v_{\text{electron}}}{m_{\text{ship}} \sqrt{1 - \dfrac{(v_{\text{electron}})^2}{c^2}}}$

$$= \frac{(9.11 \times 10^{-31}\ \cancel{\text{kg}})(0.999)(3.0 \times 10^{8}\ \text{m/s})}{(4.38 \times 10^{7}\ \cancel{\text{kg}})\sqrt{1 - (0.999)^2}}$$

$$v_{\text{ship}} = 1.39 \times 10^{-28}\ \text{m/s}$$

**Statement:** If the ship has the same momentum as the electron, its speed is $1.39 \times 10^{-28}\ \text{m/s}$. (This speed is so low that the ship would have to travel for nearly a million million years before travelling the distance of one hydrogen atom.)

**7.** No, if you were travelling on a spacecraft at 0.99*c* relative to Earth, you would not feel compressed in the direction of travel. According to the principle of relativity, experiments cannot tell us if we are at rest or moving at constant velocity. If we could get the feeling of being compressed, then we could then find some experiment that would determine this compression. As no such experiment exists, we cannot feel any changes in our bodies, including compression.

**8.** We do not notice the effects of length contraction in our everyday lives because the speeds that we experience are much less than the speed of light, so we do not notice the effects of relativistic length contraction. For example, formula-one race cars can slightly exceed 300 km/h, which is 1/1000 of the speed of light. The length contraction factor even at this high speed is $\sqrt{1-10^{-6}}$ , which is about 0.999 999 5. A car that is 5 m long at rest would shrink by only 5 μm at this speed, a distance we could only detect using a microscope. This is why cars do not appear shorter when they drive past us at high speeds.