Elastic Potential Energy and Simple Harmonic Motion

Jumping on a trampoline is fun, but is it also work? Each time the student in **Figure 1** moves downward, she has to bend her knees and push hard against the trampoline. According to Newton's third law of motion, the trampoline also exerts an equal force in the opposite direction, pushing her upward.

What type of energy does the student jumping on the trampoline have? You know that she has gravitational potential energy relative to the ground when she is in the air above the trampoline. She also has kinetic energy because she is moving. Does she have other types of energy?

As the girl pushes down on the trampoline, she stretches the elastic fabric and springs of the trampoline. The downward force of her feet does work on the trampoline, transferring energy to it. This energy temporarily becomes stored energy in the fabric and the springs. We will explore the nature of this type of stored energy in this section.

Spring Forces

One important type of potential energy is associated with springs and other elastic objects. You are probably familiar with a simple spring, such as the tight coil of wire shown in **Figure 2**. In its relaxed state, with no force applied to its end, the spring is at rest, as shown in Figure 2(a). Suppose you pull on the spring with a force F_{pull}, causing the spring to stretch to the right, as shown in Figure 2(b). When stretched, the spring exerts a force F_{spring} to the left. Likewise, if you push on the spring with a force F_{push}, it compresses to the position shown in Figure 2(c). When compressed, the spring exerts a force F_{spring} to the right. In both cases, F_{spring} is called the restorative force because it tends to restore the spring to its natural length.

Figure 1 Jumping on a trampoline requires transformations between kinetic energy, gravitational potential energy, and elastic potential energy.
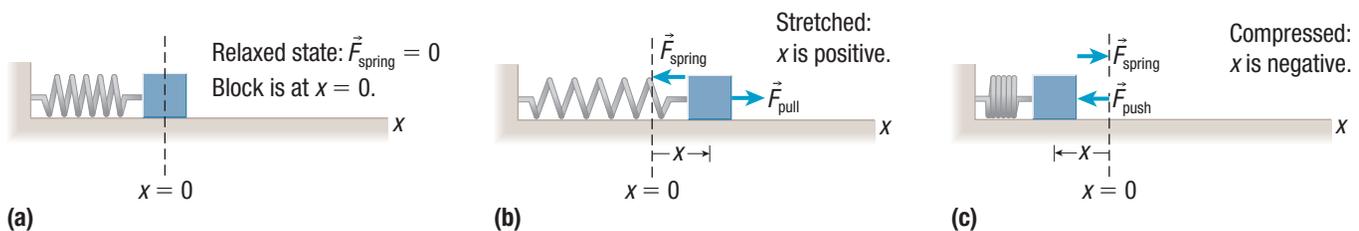

(a)

(b)

(c)

Figure 2 The force exerted by a spring is always opposite to the displacement of the end of the spring. (a) When a spring is unstretched and uncompressed, the force exerted by the spring is zero. (b) The spring is stretched by pulling it to the right, so the force exerted by the spring is to the left. (c) The spring is compressed, and the force exerted by the spring is to the right.

The amount of force exerted by a spring is proportional to the spring's displacement. This is **Hooke's law,** named after Robert Hooke (1635–1703), who discovered the relationship in 1678. Hooke's law for the force exerted by the spring is

Hooke's law the amount of force exerted by a spring is directly proportional to the displacement of the spring

$$\vec{F}_x = -k\Delta\vec{x}$$

In this equation, $\vec{F}_x$ is the force exerted by the spring on whatever stretches it, and $\Delta\vec{x}$ is the displacement of the spring from its unstretched, equilibrium position. The constant of proportionality k is called the **spring constant** of the spring, and it corresponds to the stiffness of the spring. Springs that are stiff have a larger value for k and require a larger force to extend or compress them. Springs that stretch easily have smaller values of k. An essential feature of Hooke's law is that the direction of the spring force is opposite to the direction of displacement from equilibrium. If $\Delta\vec{x}$ is upward, then $\vec{F}_x$ is downward. If $\Delta\vec{x}$ is downward, then $\vec{F}_x$ is upward.

spring constant (k) the constant of variation between the force exerted by an ideal spring and the spring's displacement

Spring Force

Skills: Performing, Observing, Analyzing, Evaluating, Communicating

SKILLS
HANDBOOK A5.5

In this activity, you will explore the force exerted by a stretched spring.

Equipment and Materials: eye protection; support stand with desk clamp; extension spring; clamp for extension spring; mass set (masses from 50 g to 200 g for a sensitive spring; masses from 500 g to 2000 g for a stiff spring); ruler; soft material

1. Put on your eye protection. Set up the equipment as shown in **Figure 3**, and place soft material below the spring.
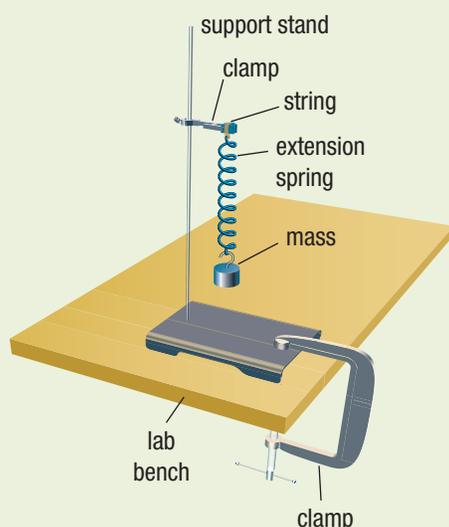

support stand

clamp

string

extension spring

mass

lab bench

clamp

Figure 3

2. Attach a mass to the spring. Measure the displacement Δx, and record your measurement in a data table. 🔴

> 🤚 Be careful not to let the masses fall on your hands or feet. Do not let the springs overstretch.

3. Repeat Step 2 for the other masses. Remember to record the masses you use.

4. For each measurement, calculate and record the force using the equation $F_g = mg$.

5. Create a graph of F_g versus Δx. Draw a line of best fit.

A. Describe the relationship between F_g and Δx. K/U T/I

B. Calculate the slope of the line of best fit. What does this slope represent? T/I A

C. Write the equation $F = k\Delta x$ for the equipment you used in this investigation, where k is the slope of the line of best fit. K/U T/I

A spring that obeys Hooke's law exactly is called an **ideal spring**, and no internal or external friction acts on it. Although we have only discussed springs so far, Hooke's law applies to many elastic devices. In Tutorial 1, you will use Hooke's law to predict the effect of an applied force on a spring.

ideal spring any spring that obeys Hooke's law; it does not experience any internal or external friction

Tutorial **1** Applying Hooke's Law

The following Sample Problem examines how to determine the spring constant of a spring and how to use the spring constant to predict the stretch of the spring when a mass is attached to it.

Sample Problem 1: Determining and Applying the Spring Constant

A spring hangs at rest from a support. If you suspend a 0.46 kg mass from the spring, its deflection is 7.9 cm (**Figure 4,** next page).

(a) Determine the spring constant.

(b) Calculate the displacement, in centimetres, of the same spring when a 0.75 kg mass hangs from it instead.

(c) Suppose the 0.75 kg mass is pushed upward, so that it rises past the spring's unstretched position, compressing the spring. Calculate the net force on the mass when the spring is compressed 5.3 cm. Include a free-body diagram.

(d) Determine the acceleration of the mass at the position given in (c) once it is released.

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Solution

(a) **Given:** $m = 0.46$ kg; $\Delta x = 7.9$ cm $= 0.079$ m
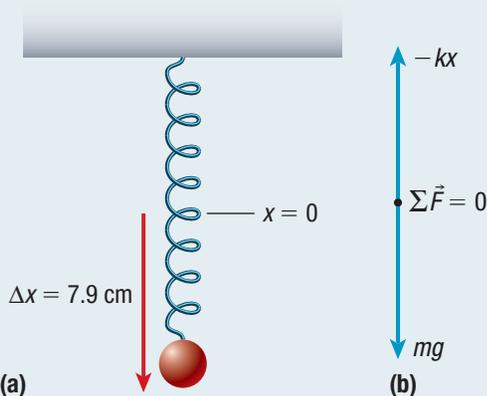
Required: k

(a)

(b)

Figure 4 (a) Mass suspended from a vertical spring (b) FBD

Analysis: The force of gravity on the mass points down. The restorative spring force on the mass points up because the spring is stretched down. To calculate the total force, subtract the magnitudes:

$$\vec{F}_g = mg\,[\text{down}] = -mg\,[\text{up}]; \vec{F}_x = -k\Delta \vec{x} = k\Delta x\,[\text{up}]$$

Since the mass is not accelerating, $\Sigma \vec{F} = 0$ according to Newton's second law.

Solution: $\Sigma \vec{F} = 0$

$$k\Delta x - mg = 0$$

$$k = \frac{mg}{\Delta x}$$

$$= \frac{(0.46\text{ kg})(9.8\text{ m/s}^2)}{(0.079\text{ m})}$$

$$k = 57.1\text{ N/m (one extra digit carried)}$$

Statement: The spring constant is 57 N/m.

(b) **Given:** $m = 0.75$ kg; $k = 57.1$ N/m

Required: $\Delta \vec{x}$

Analysis: The force of gravity on the mass points down. The spring force on the mass points up because the spring is displaced down.

$$\vec{F}_g = mg\,[\text{down}] = -mg\,[\text{up}]; \vec{F}_x = -k\Delta \vec{x} = k\Delta x\,[\text{up}]$$

Since the mass is not accelerating, $\Sigma \vec{F} = 0$.

Solution: $\Sigma \vec{F} = 0$

$$k\Delta x - mg = 0$$

$$\Delta x = \frac{mg}{k}$$

$$= \frac{(0.75\text{ kg})(9.8\text{ m/s}^2)}{57.1\text{ N/m}}$$

$$\Delta x = 0.13\text{ m}$$

The displacement is in the downward direction.

Statement: The displacement of the spring is 13 cm [down].

(c) **Given:** $m = 0.75$ kg; $k = 57.1$ N/m; $\Delta x = 5.3$ cm $= 0.053$ m

Required: $\vec{F}_{net}$

Analysis: The free-body diagram for the mass is shown in **Figure 5**.
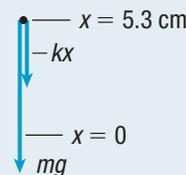

Figure 5

The force of gravity on the mass points down. The spring force on the mass points down because the spring is compressed upward.

$$\vec{F}_g = mg\,[\text{down}], \vec{F}_x = -k\Delta \vec{x} = k\Delta x\,[\text{down}]$$

$$\vec{F}_{net} = \vec{F}_g + \vec{F}_x$$

Solution: $\vec{F}_{net} = \vec{F}_g + \vec{F}_x$

$$= mg\,[\text{down}] + k\Delta x\,[\text{down}]$$

$$= (0.75\text{ kg})(9.8\text{ m/s}^2)\,[\text{down}]$$

$$+ (57.1\text{ N/m})(0.053\text{ m})\,[\text{down}]$$

$$\vec{F}_{net} = 10.4\text{ N}\,[\text{down}]$$

Statement: The net force is 10 N [down] when the spring has been compressed by 0.053 m.

(d) **Given:** $\vec{F}_{net} = 10.4$ N [down]; $m = 0.75$ kg

Required: $\vec{a}$

Analysis: $\vec{F}_{net} = m\vec{a}$

Solution: $\vec{F}_{net} = m\vec{a}$

$$\vec{a} = \frac{\vec{F}_{net}}{m}$$

$$= \frac{10.4\text{ N}}{0.75\text{ kg}}\,[\text{down}]$$

$$\vec{a} = 14\text{ m/s}^2\,[\text{down}]$$

Statement: The acceleration is 14 m/s² [down] when the spring is compressed 0.053 m. If the mass is moving upward, the downward acceleration means that it is slowing down due to the force of gravity and the elastic force of the spring.

Elastic Potential Energy

The student jumping on the trampoline in Figure 1 does work on the trampoline every time she pushes down on it. As the student comes down from a jump and hits the trampoline surface, she has kinetic energy. As the trampoline fabric and springs stretch, she transfers her kinetic energy into potential energy stored in the trampoline. Energy that is stored in objects that are compressed or stretched is called **elastic potential energy**. This stored energy in the trampoline can be transferred back to the student, giving her the kinetic energy she needs for her next upward jump.

Unlike gravitational potential energy, elastic potential energy does not depend on an object's elevation. Instead, it depends on the amount of compression or stretching. To determine the potential energy, we can calculate the change in kinetic energy of a mass attached to a spring as the spring is compressed or stretched. The change in kinetic energy equals the work done by the spring force. The work done by the spring force can be determined from a graph of applied force versus displacement.

The area under a force–displacement graph for an ideal spring has the shape of a triangle. The area of this triangle equals the work done on the spring by the applied force. This applied force is equal but opposite to the spring force on an attached object. Therefore, the work done on the spring to displace it will equal the negative of the work done by the spring on the object as it is displaced.

Figure 6 shows a graph of the spring force on a mass attached to a spring as the spring is stretched or compressed. The slope of the line equals $-k$, following Hooke's law: the direction of the spring force is always opposite to the displacement. We can interpret the area between the F versus x line and the x-axis for a given x value as the total work done on the spring as the spring stretches or compresses by Δx. This work is the negative of the total work done by the spring.

elastic potential energy the potential energy due to the stretching or compressing of an elastic material
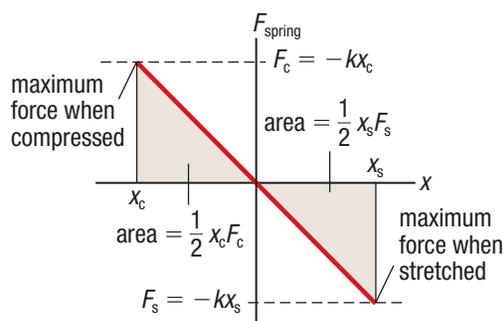

Figure 6 The work done by a variable force is equal to the area under the force–displacement graph. This figure shows the force exerted by a spring for a given displacement, which is the negative of the force applied to the spring to stretch it. For a spring, this area is a triangle. When the spring is stretched, $x = x_s$ and W is the area of the triangle at the lower right. When the spring is compressed, $x = x_c$ and W is the area of the triangle at the upper left. The work done by the spring is negative, so the change in potential energy of an attached object is positive.

Since the area of a triangle equals one-half the base length times the height, the work, W, done on a spring with a spring constant k is

$$W = \frac{1}{2}\Delta x(k\Delta x)$$

$$W = \frac{1}{2}k(\Delta x)^2$$

The work done by the spring force is the negative of this amount, and is also the negative of the change in potential energy. This means that the work done stretching or compressing the spring is transformed into elastic potential energy. Remember that work is a scalar quantity and thus directions can be ignored. We can equivalently write the equation as

$$E_e = \frac{1}{2}k(\Delta x)^2$$

where E_e is the elastic potential energy.

As with all types of energy, elastic potential energy can be transformed to kinetic energy, or to other types of potential energy. When the student jumps on the trampoline, some of the energy transforms into kinetic energy and gravitational potential energy. Some also transforms into the vibrational energy of the trampoline, sound energy, and thermal energy. The following Tutorial examines elastic potential energy.

Tutorial 2 / Calculating and Applying Elastic Potential Energy

The following Sample Problem shows how elastic potential energy can be calculated and applied in simple situations.

Sample Problem 1: Calculate Elastic Potential Energy

A 42 kg teenager balances briefly on a pogo stick, causing the spring in the stick to compress downward by 0.18 m. Determine the elastic potential energy of the teenager.

Given: $m = 42$ kg; $\Delta x = 0.18$ m

Required: E_e

Analysis: The force of gravity on the teenager points down. The spring force on the teenager points up because the spring is compressed down.

$$\vec{F}_g = mg\,[\text{down}]$$

$$\vec{F}_x = -k\Delta \vec{x}$$

$$\vec{F}_x = k\Delta x\,[\text{up}]$$

Since the teenager is not accelerating, $\Sigma\vec{F} = 0$.

Solution: Determine the spring constant:

$$\Sigma\vec{F} = 0$$

$$k\Delta x\,[\text{up}] - mg\,[\text{down}] = 0$$

$$k = \frac{mg}{\Delta x}$$

$$= \frac{(42\text{ kg})(9.8\text{ m/s}^2)}{(0.18\text{ m})}$$

$$k = 2.29 \times 10^3\text{ N/m}$$

Use the spring constant to determine the elastic potential energy:

$$E_e = \frac{1}{2}k(\Delta x)^2$$

$$= \frac{1}{2}(2.29 \times 10^3\text{ N/m})(0.18\text{ m})^2$$

$$E_e = 37\text{ J}$$

Statement: The teenager on the pogo stick has 37 J of elastic potential energy.

Practice

1. The teenager from Sample Problem 1 has a brother twice her mass. Calculate the ratio of his elastic potential energy when balancing on the pogo stick to his sister's. T/I [ans: 4:1]

2. A spring-loaded toy uses a compressed spring to fire a marble out of a tube. A force of 220 N compresses the spring by 0.14 m. Calculate the elastic potential energy of the toy. T/I [ans: 15 J]

Periodic Motion

A person usually jumps more than one time on a trampoline or a pogo stick. Often a person will jump up and down, over and over again. The motion usually changes slightly with each jump, but suppose a jumper has regular motion so that the height and time for each jump are always the same. Motion that repeats in this way is called periodic motion.

Simple Harmonic Motion

Suppose a block is connected to a spring and both are resting on a frictionless surface. The block is at equilibrium when it is resting at its initial position, $x = 0$, as shown in **Figure 7(a)**. In **Figure 7(b)**, the spring is stretched to its maximum limit, or **amplitude**, A; displacement, $x = +A$, is maximized and the block stops momentarily. The block's motion is then reversed as the spring pulls it back toward the equilibrium point (**Figure 7(c)**). The block continues to move past the equilibrium point and stops when the spring is fully compressed and negative displacement, $x = -A$, is maximized (**Figure 7(d)**). The restorative force of the spring moves the system toward equilibrium, and the cycle continues.

    Notice that the force exerted by the spring is not constant. As the spring approaches its equilibrium point, the displacement, Δx, decreases. Since F_x is proportional to the magnitude of this displacement, F_x also decreases during this time.

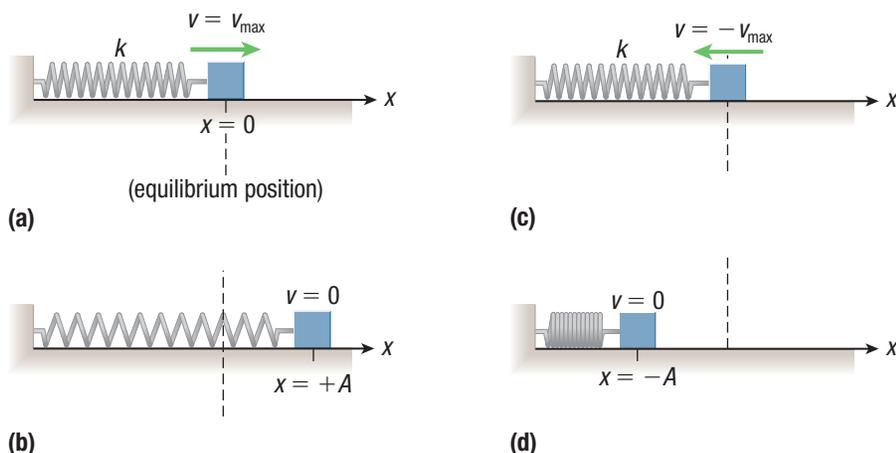<aside>amplitude (<i>A</i>) the maximum displacement of a wave</aside>

Figure 7 A mass–spring system undergoing simple harmonic motion

    The mass in Figure 7 will continue to move right and left between A and $-A$ if both the spring and surface are frictionless. This back-and-forth motion is an example of simple harmonic motion. **Simple harmonic motion** (SHM) is back-and-forth, or periodic, motion in which the moving object experiences a force that is proportional and opposite to the displacement. An object undergoing SHM is often referred to as a simple harmonic oscillator. 🌐 WEB LINK

<aside>simple harmonic motion periodic motion in which the acceleration of the moving object is proportional to its displacement</aside>

    You can visualize back-and-forth motion that is similar to SHM by thinking about a tennis ball hit from one side of the net to the other, over and over again. If the players stand still in the same positions, and if they apply the same force to the ball with each hit, the ball will continually have the same back-and-forth motion.

    When describing SHM mathematically, picture a reference circle, like the CD shown in **Figure 8** on the next page. The mass attached to the spring in the figure vibrates left and right with SHM. At the same time, the point shown on the CD rotates with uniform circular motion. Suppose that the amplitude of the SHM equals the reference point's radius of revolution. What if the period of vibration for the SHM exactly equals the period of rotation for the CD? Then, the x-coordinates of the mass and the point on the CD will remain equal at all times. This means that the acceleration of the mass is the same as the acceleration of the x-coordinate of the reference point at all times.
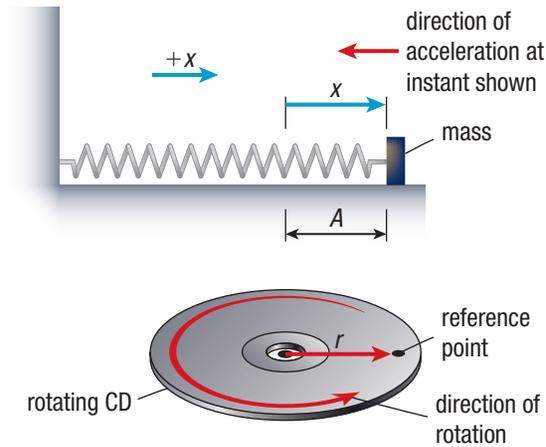
Figure 8 The CD's rotation period is the same as the mass's oscillation period. The mass and the reference point on the CD have the same *x*-coordinate at all times. The acceleration of the mass and the acceleration in the *x*-direction of the reference point are also the same at all times.

When the reference point and the mass are at the point of maximum stretch in the spring, the centripetal acceleration of the reference point a_c is directed toward negative *x*. The acceleration of the *x*-coordinate at this moment is a_c, so the acceleration of the mass at this moment is also a_c. This fact will allow us to calculate the period of motion of the mass.

For an object with radius *r* in uniform circular motion with period *T*, the centripetal acceleration is

$$a_c = \frac{4\pi^2 r}{T^2}$$

We can rewrite this equation to solve for *T*:

$$T^2 = \frac{4\pi^2 r}{a_c} \quad \text{or} \quad T = 2\pi\sqrt{\frac{r}{a_c}}$$

Applying this result to SHM, let $r = A$ (amplitude) for the reference circle. We then have

$$T = 2\pi\sqrt{\frac{A}{a_c}}$$

Next, we can use Hooke's law and Newton's second law to calculate the acceleration of the mass. The equation

$$\vec{F}_x = -k\Delta\vec{x} = m\vec{a}_x$$

means that

$$\vec{a}_x = -\frac{k\Delta\vec{x}}{m}$$

The ratio of the magnitude of the displacement to the magnitude of the acceleration is

$$\frac{\Delta x}{a_x} = \frac{m}{k}$$

At the point of maximum stretch,

$$\frac{\Delta x}{a_x} = \frac{A}{a_c}$$

which means

$$\frac{m}{k} = \frac{A}{a_c}$$

If you substitute this result into the equation for the period, we obtain the equation of the period of a mass on a spring:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

The period, T, of SHM is the amount of time for one cycle of motion. Period is measured in seconds. The inverse of T equals the number of cycles per second, or the frequency, f. The units of frequency are called hertz, and 1 Hz = 1 cycle/s. ⊕ WEB LINK

In Tutorial 3, you will use the equations for simple harmonic motion, period, and frequency to solve problems.

Tutorial 3 Application of Period and Frequency

The following Sample Problem shows how to calculate the period and frequency of a diver standing on a diving board.

Sample Problem 1: Calculating Period and Frequency

When a diving board bends, the restoring force due to the board's elastic properties obeys Hooke's law for an elastic material. Suppose a diver of mass 85 kg stands on a diving board with spring constant 8.1×10^3 N/m. The mass of the board is much smaller than the diver's mass. Calculate the period and frequency at which the board vibrates.

Given: $m = 85$ kg; $k = 8.1 \times 10^3$ N/m

Required: f, T

Analysis: Use the equations for simple harmonic motion period and frequency:

$$T = 2\pi\sqrt{\frac{m}{k}} \text{ and } f = \frac{1}{T}$$

Solution: $T = 2\pi\sqrt{\frac{m}{k}}$

$$= 2\pi\sqrt{\frac{85 \text{ kg}}{8.1 \times 10^3 \text{ N/m}}}$$

$$T = 0.64 \text{ s}$$

$$f = \frac{1}{T}$$

$$= \frac{1}{0.64 \text{ s}}$$

$$f = 1.6 \text{ Hz}$$

Statement: The diving board will vibrate with a period of 0.64 s and a frequency of 1.6 Hz.

Practice

1. A 105 kg swimmer stands on a diving board with a spring constant of 7.6×10^3 N/m. Determine the period and frequency of the board vibrations. T/I [ans: 0.74 s; 1.4 Hz]

2. A car mounted on the springs in its suspension acts like a mass on a spring. How will the frequency of oscillations change if passengers are added to the car? Will the frequency increase, decrease, or stay the same? Explain your answer. K/U T/I A

Notice that the calculations in Tutorial 3 did not require you to identify how far the diving board bent. The period and frequency depend only on the mass that exerts the force and the spring constant of the elastic material.

You can identify numerous examples of SHM in everyday life (if you ignore friction). A child on a playground swing may swing back and forth at a regular rate. Strings on guitars and violins vibrate when plucked. The planets move in periodic orbits around the Sun. Automobiles use springs to provide a cushioning effect. However, you do not experience long periods of SHM when riding in a car because shock absorbers provide friction on the springs. We will explore the behaviour of shock absorbers further in Section 4.7.

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Summary

- Hooke's law states that the force exerted by a spring (or, equivalently, the force applied to a spring) is directly proportional to the spring's displacement from its rest equilibrium position, $\vec{F} = -k\Delta\vec{x}$.
- The force exerted by a spring is a restorative force. It acts in the opposite direction of the displacement to return the spring to its natural length.
- The constant of proportionality in Hooke's law is the spring constant, k. The spring constant is large when a spring is stiff and small when a spring is loose. The spring constant is measured in newtons per metre.
- The energy stored in an object that is stretched, compressed, twisted, or bent is called elastic potential energy, $E_e = \frac{1}{2}k(\Delta x)^2$.
- Simple harmonic motion (SHM) is periodic motion in which an object moves in response to a force that is directly proportional and opposite to its displacement.

Questions

1. Spring A has a spring constant of 70 N/m, and spring B has a spring constant of 50 N/m. Explain which spring is more difficult to stretch. K/U

2. A force of 5 N is applied to a block attached to the free end of a spring stretched from its relaxed length by 10 mm. Determine the spring constant. T/I

3. Is the elastic potential energy stored in a spring greater when the spring is stretched by 1.5 cm or when it is compressed by 1.5 cm? Explain your answer. K/U C

4. Calculate the elastic potential energy stored in a spring with a spring constant of 5.5×10^3 N/m when it
   (a) stretches 2.0 cm
   (b) compresses 3.0 cm K/U T/I

5. A 0.63 kg mass rests on top of a vertical spring with spring constant 65 N/m. T/I
   (a) When the mass sits at rest, determine the distance that the spring is compressed from its equilibrium position.
   (b) The mass is held at the unstretched position of the spring and released. Calculate the acceleration of the mass after it falls 4.1 cm.

6. A 5.2 kg mass hung from a spring vibrates with a period of 1.2 s. Calculate the spring constant. T/I

7. A spring has a spring constant of 1.5×10^3 N/m. Determine the length that the spring should be stretched to store 80.0 J of energy. T/I

8. Calculate the work done by a spring force acting on a spring attached to a box, stretched from its relaxed length by 15 mm. The spring constant of the spring is 400.0 N/m. T/I

9. A mass–spring system undergoes SHM. The elastic potential energy at maximum stretch is 7.50 J, the mass is 0.20 kg, and the spring constant is 240 N/m. Calculate the frequency and amplitude of oscillation. T/I

10. The springs in the suspension of a car with worn-out shock absorbers will undergo SHM after hitting a bump in the road. Suppose that a car with worn-out shock absorbers has two identical rear axle springs that each support 5.5×10^2 kg. After hitting a large pothole, the rear end of the car vibrates through six cycles in 4.4 s. Calculate the spring constant of either spring. T/I A

11. Pyon pyon "flying shoes" were invented by Yoshiro Nakamatsu of Japan (**Figure 9**). Research this unique invention. Draw a diagram showing the forces at work when the wearer takes a step. How do you think the shoe's designer incorporated Hooke's law into the shoe design? 🌐 K/U A

Figure 9

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