3.2 Centripetal Acceleration

The hammer throw is a track-and-field event in which an athlete throws a "hammer"—a heavy metal ball attached to a wire and handle—the farthest distance possible (**Figure 1**). To do this, the athlete first swings the ball in a circular path. After giving it a maximum circular speed with three or four turns, the athlete releases the hammer, which then flies down the field. Just before the hammer is released, it has **uniform circular motion**, which is motion in a circular path at a constant speed.

uniform circular motion the motion of an object with a constant speed along a circular path of constant radius

Figure 1 To give the hammer enough speed to travel a long distance down the field, the athlete must move it rapidly in a circular path.

By moving the ball in a circle, the athlete introduces the force of tension in the wire. This tension keeps the ball in a circular path. The greater the tension, the greater the acceleration toward the centre of the circle and the faster the ball moves in a circular path. When the tension is very large, so is the speed of the ball. When the athlete releases the hammer, the ball travels far down the field.

You may not always realize it, but objects moving in circular paths are all around you. Clothes in a washing machine during the spin cycle, the drum of a clothes dryer, the hands of certain electric clocks, and the spinning blades of a blender and a lawn mower: all of these objects move with uniform circular motion. What you may not have considered is that these are all among the most common non-inertial frames of reference. These objects move in a circular path, so their velocity constantly changes direction. Therefore they are accelerating. Acceleration that is directed toward the centre of a circular path is called **centripetal acceleration**, $\vec{a}_c$.

centripetal acceleration ($\vec{a}_c$) the instantaneous acceleration that is directed toward the centre of a circular path

Equations for Centripetal Acceleration

Recall that the average acceleration, $\vec{a}_{av}$, of an object equals the change in velocity, $\Delta\vec{v}$, during an interval of time, Δt: $\vec{a}_{av} = \frac{\Delta\vec{v}}{\Delta t}$. For an object moving with uniform circular motion, the velocity changes direction continuously with time, so $\Delta\vec{v}$ is definitely not zero. Therefore, centripetal acceleration is not zero.

To calculate centripetal acceleration, we now consider the example of a runner moving at a constant speed along a circular track. The velocity of the runner changes with time and is always tangential to the circular path, as shown in **Figure 2**. Figure 2(a) shows a runner's velocity vectors at two nearby positions: A′ and A. Figure 2(b) shows the corresponding change in velocity, $\Delta\vec{v}$, over a short time interval, Δt. Recall from Section 1.4 that the difference in velocity vectors is the same as adding one vector to the negative of the other vector. First, shift vector $\vec{v}_2$ down so that its head is at point A (Figure 2(b)). Then reverse $\vec{v}_1$ so $\vec{v}_1$ becomes $-\vec{v}_1$. Place the tail of vector $-\vec{v}_1$ at the head of vector $\vec{v}_2$, so that the sum of the vectors is $\Delta\vec{v}$, which points toward the centre of the circle. It should be noted that we actually need to decrease the size of the time interval until it is very small for $\Delta\vec{v}$ to point directly toward the centre. For the sake of discussion and illustrating the concept, however, our model in Figure 2 is satisfactory.

As you can see in Figure 2(a), the individual velocity vectors $\vec{v}_1$ and $\vec{v}_2$ are both tangent to the circle, perpendicular to the circle's radius, and equal in length (magnitude). This is true for all of the runner's velocity vectors along the circular path. The acceleration vector has the same direction as $\Delta\vec{v}$, so it follows that the centripetal acceleration of an object must always point toward the centre of the circular path.

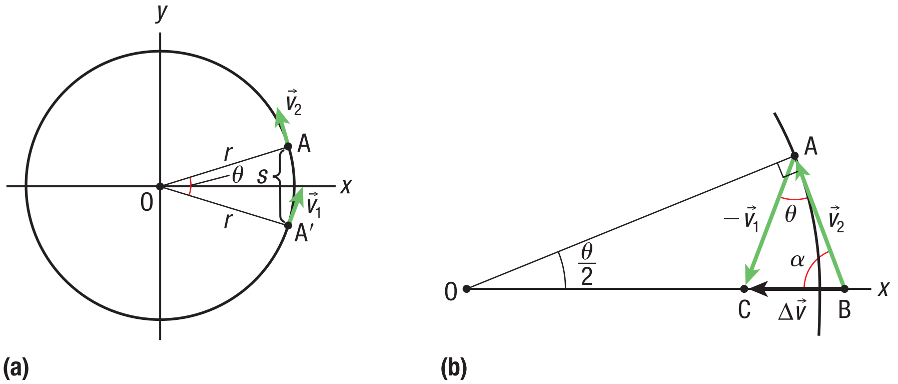

Figure 2 (a) The velocity $\vec{v}$ of an object (in this case, a runner) moving with uniform circular motion is shown as $\vec{v}_1$ and $\vec{v}_2$ at two different locations along the circular path. The distance travelled in going from point A′ to point A is *s*. (b) The difference in the velocity vectors, $\Delta\vec{v}$, is directed toward the centre of the circle when the time interval is very small.

Now use the triangle BAC in Figure 2(b) to calculate the magnitude of the acceleration. This triangle has two sides with equal lengths, $|\vec{v}_1|$ and $|\vec{v}_2|$. In general, the velocity magnitude is the same at any point around the circle, so $|\vec{v}_1| = |\vec{v}_2| = |\vec{v}|$, or simply v. The third side of this triangle is the vector $\Delta\vec{v}$, which has a length of $|\Delta\vec{v}|$.

The triangle BAC has the same interior angles as triangle AOA′ in Figure 2(a), so these triangles are similar. You can check this by using the relations below to show that the angle θ is the same for both triangles. Note in Figure 2(b) that the angle at point A between $\vec{v}_2$ and the radius is 90°, so

$$\frac{\theta}{2} + \alpha = 90^\circ$$
$$\theta + 2\alpha = 180^\circ$$
$$\theta = 180^\circ - 2\alpha$$

When you add the two equal angles, α, within triangle BAC to the third angle, they equal 180°, so the third angle must equal

$$180 - 2\alpha = \theta$$

Two sides of triangle AOA′ are along the radius of the circle, so they have length r, while the other side (between points A′ and A) has length s. When Δt is small, the arc length between points A′ and A approaches a straight-line length that connects A′ and A. Therefore, the distance the runner travels from A′ to A is approximately equal to the distance given by $s \approx v\Delta t$. The triangles BAC and AOA′ are similar, so the ratios of their corresponding sides are equal. Substituting v for the magnitude of either $\vec{v}_1$ or $\vec{v}_2$, and assuming a very small Δt:

$$\frac{|\Delta\vec{v}|}{v} = \frac{s}{r}$$
$$\frac{|\Delta\vec{v}|}{v} = \frac{v\Delta t}{r}$$
$$|\Delta\vec{v}| = \frac{v^2\Delta t}{r}$$

The magnitude of the average acceleration equals the magnitude of the difference in the velocities ($|\Delta\vec{v}|$) divided by Δt:

$$a_{av} = \frac{|\Delta\vec{v}|}{\Delta t}$$

$$= \frac{\frac{v^2 \Delta t}{r}}{\Delta t}$$

$$a_{av} = \frac{v^2}{r}$$

In the above equation, $a_{av} = a_c$ when the time interval is very small:

$$a_c = \frac{v^2}{r}$$

where a_c is the magnitude of the centripetal acceleration, v is the speed of the object moving along the circular path, and r is the radius of the circular path. Note that, although this derivation started with the definition of average acceleration, the result becomes exact for a very small time interval Δt, so the centripetal acceleration in this case is an instantaneous quantity directed toward the centre of the circle.

The equation for centripetal acceleration indicates that, when the speed of an object moving with uniform circular motion is large for a constant radius, such as in the case of the hammer in the hammer throw, the direction of the velocity changes more rapidly than it would for a smaller speed. This means that, to produce these rapid changes in velocity, you need a larger acceleration. When the radius is larger for a constant speed, the direction of the velocity changes more slowly, so the object has a smaller acceleration.

period (*T*) the time required for a rotating, revolving, or vibrating object to complete one cycle

Sometimes you may not know the speed of an object moving with uniform circular motion. However, you may be able to measure the time it takes for the object to move once around the circle, or the **period**, T. Then you can calculate the speed. Remember that the speed is constant, and that it equals the length of the path the object travels (the circumference of the circle, or $2\pi r$) divided by the period, T:

$$v = \frac{\Delta d}{\Delta t}$$

$\Delta d = 2\pi r$ and $\Delta t = T$, so

$$v = \frac{2\pi r}{T}$$

Substitute the above expression for v into the above equation for centripetal acceleration to obtain the acceleration in terms of the period and the radius:

$$a_c = \frac{v^2}{r}$$

$$= \frac{\left(\frac{2\pi r}{T}\right)^2}{r}$$

$$= \frac{\frac{4\pi^2 r^2}{T^2}}{r}$$

$$a_c = \frac{4\pi^2 r}{T^2}$$

For high rotational speeds, frequency is the preferred quantity of measurement. The **frequency**, f, equals the number of revolutions per unit of time, or

$$f = \frac{1}{T}$$

The unit of frequency is hertz (Hz), or cycles per second.

In terms of frequency and radius, the equation for centripetal acceleration takes the form

$$\begin{aligned} a_c &= \frac{4\pi^2 r}{T^2} \\ &= \frac{4\pi^2 r}{\left(\frac{1}{f}\right)^2} \\ &= \frac{4\pi^2 r}{\frac{1}{f^2}} \end{aligned}$$

$$a_c = 4\pi^2 r f^2$$

You now have three equations for determining the magnitude of the centripetal acceleration. When dealing with the vector of this acceleration, remember that centripetal acceleration always points toward the centre of the circle. The following Tutorial models how to solve problems involving centripetal acceleration.

frequency (f) the number of rotations, revolutions, or vibrations of an object per unit of time; the inverse of period; SI unit Hz

UNIT TASK BOOKMARK

You can use some of the equations for centripetal acceleration when you complete the Unit Task on page 146.

Tutorial 1 Solving Problems with Objects Moving with Centripetal Acceleration

This Tutorial shows how to calculate the centripetal acceleration for an object undergoing uniform circular motion using the different equations for the magnitude of centripetal acceleration.

Sample Problem 1: Calculating the Magnitude of Centripetal Acceleration

A child rides a carousel with a radius of 5.1 m that rotates with a constant speed of 2.2 m/s. Calculate the magnitude of the centripetal acceleration of the child.

Given: $r = 5.1$ m; $v = 2.2$ m/s

Required: a_c

Analysis: $a_c = \frac{v^2}{r}$

Solution:

$$\begin{aligned} a_c &= \frac{v^2}{r} \\ &= \frac{(2.2 \text{ m/s})^2}{5.1 \text{ m}} \\ a_c &= 0.95 \text{ m/s}^2 \end{aligned}$$

Statement: The magnitude of the centripetal acceleration of the child is 0.95 m/s^2.

Sample Problem 2: Calculating the Magnitude and Direction of Centripetal Acceleration

A salad spinner with a radius of 9.7 cm rotates clockwise with a frequency of 12 Hz. At a given instant, the lettuce in the spinner moves in the westward direction (**Figure 3**). Determine the magnitude and direction of the centripetal acceleration of the piece of lettuce in the salad spinner at the moment shown in Figure 3.

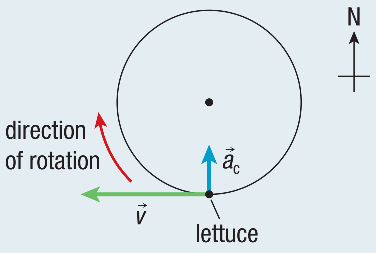

Figure 3

Given: $r = 9.7$ cm $= 0.097$ m; $f = 12$ Hz

Required: $\vec{a}_c$

Analysis: First, determine the direction of the acceleration from Figure 3. Then calculate the magnitude of the acceleration using the equation $a_c = 4\pi^2 rf^2$.

Solution: The westward velocity vector is at the south end of the spinner, as Figure 3 indicates. The direction of the centripetal acceleration is north.

$$a_c = 4\pi^2 rf^2$$
$$= 4\pi^2(0.097 \text{ m})(12 \text{ Hz})^2$$
$$a_c = 5.5 \times 10^2 \text{ m/s}^2$$

Statement: The centripetal acceleration of the lettuce at the moment shown in Figure 3 is 5.5×10^2 m/s^2 [N].

Sample Problem 3: Calculating Frequency and Period of Rotation for a Spinning Object

The centripetal acceleration at the end of an electric fan blade has a magnitude of 1.75×10^3 m/s^2. The distance between the tip of the fan blade and the centre is 12 cm. Calculate the frequency and the period of rotation of the fan.

Given: $a_c = 1.75 \times 10^3$ m/s^2; $r = 12$ cm $= 0.12$ m

Required: f; T

Analysis: Use the equation for centripetal acceleration that includes frequency and radius: $a_c = 4\pi^2 rf^2$; rearrange and solve for f. Then use the equation relating frequency and period to calculate the period of rotation: $T = \frac{1}{f}$.

$$a_c = 4\pi^2 rf^2$$
$$\frac{a_c}{4\pi^2 r} = f^2$$
$$f = \sqrt{\frac{a_c}{4\pi^2 r}}$$

Solution: $f = \sqrt{\frac{a_c}{4\pi^2 r}}$

$$= \sqrt{\frac{1.75 \times 10^3 \text{ m/s}^2}{4\pi^2(0.12 \text{ m})}}$$
$$= \pm 19.2 \text{ Hz}$$

Choose the positive root because frequency cannot be negative.

$$f = 19.2 \text{ Hz} \ \text{(one extra digit carried)}$$
$$T = \frac{1}{f}$$
$$= \frac{1}{19.2 \text{ Hz}}$$
$$T = 5.2 \times 10^{-2} \text{ s}$$

Statement: The frequency of the fan is 19 Hz, and the period of rotation is 5.2×10^{-2} s.

Practice

1. At a distance of 25 km from the eye (centre) of a hurricane, the wind moves at nearly 50.0 m/s. Assume that the wind moves in a circular path. Calculate the magnitude of the centripetal acceleration of the particles in the wind at this distance. T/I A [ans: 0.10 m/s^2]
2. An athlete in a hammer throw competition swings the hammer with uniform circular motion clockwise as viewed from above at a speed of 4.24 m/s and a distance of 1.2 m from the centre of the circle. At a given instant, the hammer's velocity is directed southward. Determine the centripetal acceleration at this instant. T/I [ans: 15 m/s^2 [W]]
3. A ball on a string moves in a horizontal circle of radius 1.4 m. The centripetal acceleration of the ball has a magnitude of 12 m/s^2. Calculate the speed of the ball. T/I A [ans: 4.1 m/s]
4. The planet Venus moves in a nearly circular orbit around the Sun. The average radius of its orbit is 1.08×10^{11} m. The centripetal acceleration of Venus has a magnitude of 1.12×10^{-2} m/s^2. Calculate Venus's period of revolution around the Sun (a) in seconds and (b) in Earth days. T/I A [ans: (a) 1.95×10^7 s; (b) 226 days]
5. Suppose a satellite revolves around Earth in a circular orbit. The speed of the satellite is 7.27×10^3 m/s, and the radius of its orbit, with respect to Earth's centre, is 7.54×10^6 m. Calculate the magnitude of the satellite's centripetal acceleration. T/I A [ans: 7.01 m/s^2]
6. A research apparatus called a centrifuge undergoes centripetal acceleration with a magnitude of 3.3×10^6 m/s^2. The centrifuge has a radius of 8.4 cm. Calculate the frequency of the centrifuge (a) in hertz and (b) in revolutions per minute (rpm). T/I A [ans: (a) 1.0×10^4 Hz; (b) 6.0×10^5 rpm]

3.2 Review

Summary

- Uniform circular motion is the motion of any body that follows a circular path at a constant speed.
- Centripetal acceleration is the instantaneous acceleration of an object toward the centre of a circular path.
- There are three equations to determine centripetal acceleration: $a_c = \frac{v^2}{r}$, $a_c = \frac{4\pi^2 r}{T^2}$, and $a_c = 4\pi^2 r f^2$.

Questions

1. You have a puck on a string, and you twirl the puck with uniform circular motion in a horizontal circle along virtually frictionless ice. K/U T/I A
   (a) What causes the centripetal acceleration of the puck?
   (b) How does doubling the radius of the circle and leaving the speed unchanged affect the centripetal acceleration?
   (c) How does doubling the speed and leaving the radius unchanged affect the centripetal acceleration?
2. Two athletes compete in the hammer throw. One athlete can spin the hammer twice as fast as the second athlete. Compare the magnitudes of the two centripetal accelerations for the two hammer throws. Explain your answer. T/I C A
3. In a rodeo, a performer twirls a lasso (rope) at a constant speed, and the lasso turns in a circle of radius 0.42 m. The lasso has a period of rotation of 1.5 s. Calculate the magnitude of the centripetal acceleration of the lasso. T/I A
4. A motorcyclist maintains a constant speed of 28 m/s while racing on a circular track with a constant radius of 135 m. Calculate the magnitude of the centripetal acceleration of the motorcyclist. T/I A
5. The centripetal acceleration of an object at Earth's equator results from the daily rotation of Earth. Calculate the object's centripetal acceleration, given that the radius of Earth at the equator is 6.38×10^6 m. T/I A
6. An amusement park ride consists of a rotating cylinder with a coarse fabric on the walls, for friction. Participants on this ride stand against the wall as the cylinder rotates. After the cylinder reaches a constant speed, the floor of the ride drops away beneath the occupants. They remain against the wall because of the centripetal acceleration, which must be greater than about 25 m/s^2. This ride has a radius of 2.0 m. Determine the minimum frequency of rotation of the cylinder. T/I A
7. The centripetal acceleration of a car moving around a circular curve at a constant speed of 22 m/s has a magnitude of 7.8 m/s^2. Calculate the radius of the curve. T/I A
8. A jogger is running around a circular track that has a circumference of 478 m. The magnitude of the centripetal acceleration of the jogger is 0.146 m/s^2. Calculate the jogger's speed in kilometres per hour. T/I A
9. A bicycle wheel with a radius of 0.300 m is spinning clockwise at a rate of 60.0 rpm. T/I A
   (a) Calculate the period of the wheel's motion.
   (b) Calculate the centripetal acceleration of a point on the edge of the wheel if at that instant it moves westward.
10. The Moon's period of revolution is 27.3 days, and the magnitude of its centripetal acceleration is about 2.7×10^{-3} m/s^2. T/I A
    (a) Calculate the distance between the centre of the Moon and the centre of Earth. Assume that the orbit of the Moon is circular and that its speed is constant.
    (b) Compare your answer with the value provided in Appendix B. If different, suggest reasons why.
11. The record distance for the hammer throw is about 87 m. To achieve this distance, an athlete must produce a centripetal acceleration of nearly 711 m/s^2. K/U T/I A
    (a) Given a radius of 1.21 m, calculate the speed of the ball when it is released.
    (b) The athlete lets go when the ball is 2.0 m above the ground and moving at an angle of 42° above the horizontal. Determine the range. Ignore any air friction.