1.5 Projectile Motion

In sports in which a player kicks, throws, or hits a ball across a field or court, the player's initial contact with the ball propels the ball upward at an angle. The ball rises to a certain point, and gravity eventually curves the path of the ball downward. If you ignore the effects of air resistance and Earth's rotation, the curved path, or trajectory, of the ball under the influence of Earth's gravity follows the curve of a parabola, as **Figure 1** shows. The ball acts like a **projectile**, which is an object that is moving through the air and accelerating due to gravity. The *x*-direction is horizontal and positive to the right, and the *y*-direction is vertical and positive upward.

projectile an object that is launched through the air along a parabolic trajectory and accelerates due to gravity

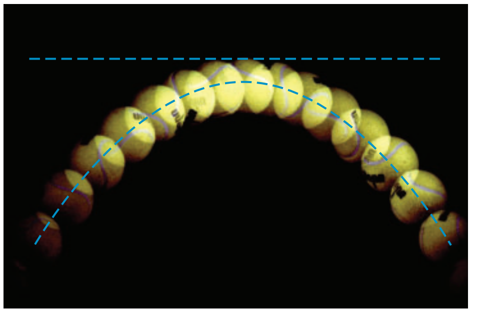


Figure 1 The path of a projectile follows the curve of a parabola.

The ball in Figure 1 was hit with a tennis racquet. If you draw an imaginary line through the ball images, you can trace the parabola from where the ball made contact with the racquet to the other end. Another imaginary line shows the uppermost point of the trajectory (at the top of the highest ball). After the ball leaves the racquet, its path curves upward to this highest point and then curves downward. You can see the symmetry of the ball's path because the shape of the upward-bound curve exactly matches the shape of the downward-bound curve. Anyone who has tossed any kind of object into the air has observed this parabolic trajectory called projectile motion. Before we formally define projectile motion, we will look at its properties.

Properties of Projectile Motion

Suppose you drop a soccer ball from the roof of a one-storey building while your friend stands next to you and kicks another soccer ball horizontally at the same instant. Will they both land at the same time? Some people are surprised to learn that the answer is yes.

Figure 2, on the next page, shows a strobe image of two balls released simultaneously, one with a horizontal projection, as your friend's soccer ball had. The horizontal lines represent equal time intervals—the time interval between the camera's strobe flashes is constant. The vertical components of the displacement increase by the same amount for each ball. The horizontal displacement of the projectile—in this instance, the ball—is called the horizontal **range**, Δd_x. The horizontal motion is also constant. The trajectory forms from the combination of the independent horizontal and vertical motions.

range (Δd_x) the horizontal displacement of a projectile

We observe the following properties about the motion of a projectile:

- The horizontal motion of a projectile is constant.
- The horizontal component of acceleration of a projectile is zero.
- The vertical acceleration of a projectile is constant because of gravity.
- The horizontal and vertical motions of a projectile are independent, but they share the same time.

Combining these properties helps us define projectile motion: **projectile motion** is the motion of an object such that the horizontal component of the velocity is constant and the vertical motion has a constant acceleration due to gravity.

projectile motion the motion of a projectile such that the horizontal component of the velocity is constant, and the vertical motion has a constant acceleration due to gravity

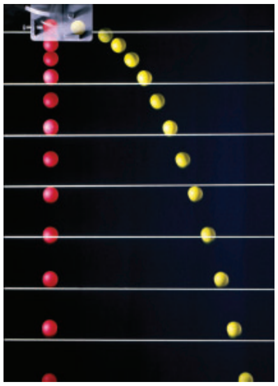


Figure 2 The two balls reach the lowest position at the same instant, even though one ball was dropped and the other was given an initial horizontal velocity.

The most important property of projectile motion in two dimensions is that the horizontal and vertical motions are completely independent of each other. This means that motion in one direction has no effect on motion in the other direction. This allows us to separate a complex two-dimensional projectile motion problem into two separate simple problems: one that involves horizontal, uniform motion and one that involves vertical, uniform acceleration down. **Figure 3** shows a baseball player hitting a fly ball and the path it follows. You can see that the horizontal velocity v_x is independent of the vertical velocity v_y.

Investigation **1.5.1**

Investigating Projectile Motion (page 50)
In this investigation, you will use an air table to investigate projectile motion.

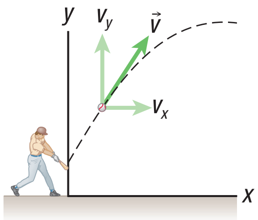


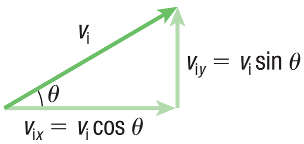
Figure 3 The horizontal and vertical components of velocity are independent of each other.

You can also see this result in the strobe images in Figure 2. The ball on the left simply drops, but the ball on the right has an initial horizontal velocity. The ball on the left falls straight down, while the ball on the right follows a parabolic path typical of projectile motion. The balls have quite different horizontal velocities at each "flashpoint" in the image. Nonetheless, they are at identical heights at each point. This shows that their displacements and velocities along the y-direction are the same. The image confirms that the motion along the vertical direction does not depend on the motion along the horizontal direction. WEB LINK

Analyzing Projectile Motion

In Section 1.2, you reviewed the equations that describe motion in one dimension. You can use these same equations to analyze the motion of a projectile in two dimensions. You simply have to apply the equations to the x- and y-motions separately. Assume that at $t = 0$ the projectile leaves the origin with an initial velocity v_i. If the velocity vector makes an angle θ with the horizontal, where θ is the projection angle, then from the definitions of the cosine and sine functions,

$$v_{ix} = v_i \cos\theta$$
$$v_{iy} = v_i \sin\theta$$



where v_{ix} is the initial velocity (at $t = 0$) in the x-direction, and v_{iy} is the initial velocity in the y-direction.

Table 1 summarizes the kinematics equations you can use with both horizontal and vertical components.

Table 1 Kinematics Equations with Horizontal and Vertical Components

Direction of motion	Description	Equations of motion
horizontal motion (*x*)	constant-velocity equation for the *x*-component only	$v_{ix} = v_i \cos\theta$ $v_{ix} = \text{constant}$ $\Delta d_x = v_{ix}\Delta t$ $\Delta d_x = (v_i \cos\theta)\Delta t$
vertical motion (*y*)	constant-acceleration equations for the *y*-component; constant acceleration has a magnitude of $\lvert \vec{g} \rvert = g = 9.8 \text{ m/s}^2$	$v_{fy} = v_i \sin\theta - g\Delta t$ $\Delta d_y = (v_i \sin\theta)\Delta t - \frac{1}{2}g\Delta t^2$ $v_{fy}^2 = (v_i \sin\theta)^2 - 2g\Delta d_y$

Mini Investigation

Analyzing the Range of a Projectile

Skills: Performing, Analyzing, Communicating

SKILLS HANDBOOK A2.2

You can calculate the horizontal range of a projectile by applying the kinematics equations step by step. In this activity, you will complete a table showing launch angle, time of flight, maximum height, and range.

Equipment and Materials: paper and pencil; calculator

1. Set up a table like the one in **Table 2**, either on paper or electronically.

Table 2

Launch angle (θ)	Time of flight (s)	Maximum height (m)	Range (m)
5			
15			
25			
85			

2. List several launch angles in increments of 10°, from 5° to 85°.
3. Complete the table for a projectile that has an initial velocity of magnitude 25 m/s and lands at the same level from which it was launched. Use two significant digits in your calculations.

A. What conclusion can you draw from the data about the relationship between the horizontal component of velocity and maximum height? K/U T/I

B. What conclusion can you draw from the data about how you can maximize the height of an object in projectile motion? K/U T/I

C. What conclusion can you draw from the data about how you can maximize the range of an object in projectile motion? K/U T/I

D. The sum of complementary angles is 90°. Identify pairs of complementary angles. Look at the range for each pair of complementary angles in your data. Write a statement that summarizes the relationship between complementary initial angles for projectile motion. K/U T/I C

In the following Tutorial, you will apply the projectile motion equations to Sample Problems in which an object launches horizontally and an object launches at an angle above the horizontal.

Tutorial 1 Solving Simple Projectile Motion Problems

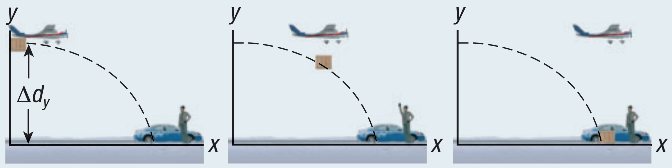
This Tutorial demonstrates how to solve two-dimensional projectile motion problems. In Sample Problem 1, an object launches horizontally so that it has an initial horizontal velocity but no initial vertical velocity. In Sample Problem 2, an object launches at an upward angle so that it has both initial horizontal and vertical velocity components.

Sample Problem 1: Solving Projectile Motion Problems with No Initial Vertical Velocity

An airplane carries relief supplies to a motorist stranded in a snowstorm. The pilot cannot safely land, so he has to drop the package of supplies as he flies horizontally at a height of 350 m over the highway. The speed of the airplane is a constant 52 m/s. **Figure 4** shows the package (a) as it leaves the airplane, (b) in mid-drop, and (c) when it lands on the highway.

(a) Calculate how long it takes for the package to reach the highway.

(b) Determine the range of the package.



(a) **(b)** **(c)**

Figure 4

Solution

(a) **Given:** $\Delta d_y = -350$ m; $v_i = 52$ m/s

Required: Δt

Analysis: Set $d_i = 0$ as the altitude at which the plane is flying. Therefore, $\Delta d_y = -350$ m. Calculate Δt from the formula for the displacement along y:

$$\Delta d_y = (v_i \sin\theta)\Delta t - \frac{1}{2}g\Delta t^2$$

Solution:

$$\begin{aligned}\Delta d_y &= (v_i \sin\theta)\Delta t - \frac{1}{2}g\Delta t^2\\ &= (52\text{ m/s})(\sin 0°)\Delta t - \frac{1}{2}g\Delta t^2\\ &= (0)\Delta t - \frac{1}{2}g\Delta t^2\\ \Delta d_y &= -\frac{1}{2}g\Delta t^2\\ \Delta t &= \sqrt{\frac{-2\Delta d_y}{g}}\\ &= \sqrt{\frac{-2(-350\text{ m})}{9.8\text{ m/s}^2}}\\ \Delta t &= 8.45\text{ s (one extra digit carried)}\end{aligned}$$

Statement: The package takes 8.5 s to reach the highway.

(b) **Given:** $\Delta d_y = -350$ m; $v_i = 52$ m/s; $\Delta t = 8.45$ s

Required: Δd_x

Analysis: Calculate Δd_x using the definition of cosine: $\Delta d_x = (v_i \cos\theta)\Delta t$

Solution:

$$\begin{aligned}\Delta d_x &= (v_i \cos\theta)\Delta t\\ &= (52\text{ m/s})(\cos 0°)(8.45\text{ s})\\ \Delta d_x &= 4.4 \times 10^2\text{ m}\end{aligned}$$

Statement: The range of the package is 4.4×10^2 m.

Sample Problem 2: Solving Projectile Motion Problems with an Initial Vertical Velocity

A golfer hits a golf ball with an initial velocity of 25 m/s at an angle of 30.0° above the horizontal. The golfer is at an initial height of 14 m above the point where the ball lands (**Figure 5**).

(a) Calculate the maximum height of the ball.

(b) Determine the ball's velocity on landing.

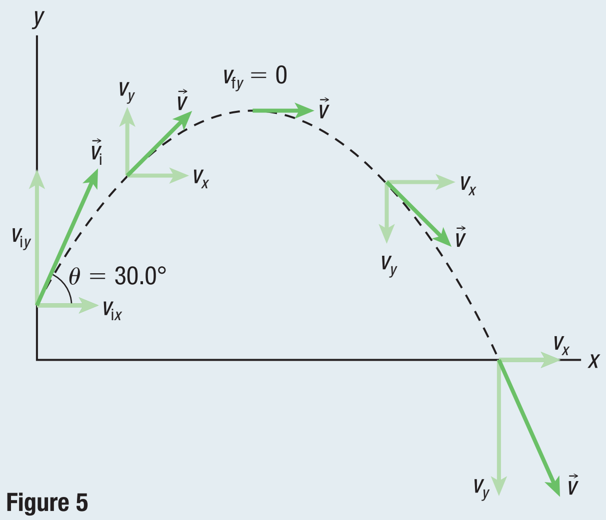


Figure 5

Solution

(a) **Given:** $v_i = 25$ m/s; $\theta = 30.0°$

Required: $\Delta d_{y\,max}$

Analysis: When the golf ball reaches its maximum height, the y-component of the ball's velocity is zero. So $v_{fy} = 0$. Set the formula for vertical velocity, $v_{fy} = v_i \sin\theta - g\Delta t$, equal to zero, and then determine the time at which the ball reaches this point. Then, determine the maximum height using the formula for vertical displacement,

$$\Delta d_y = (v_i \sin\theta)\Delta t - \frac{1}{2}g\Delta t^2.$$

Solution:

$$v_{fy} = v_i \sin\theta - g\Delta t$$
$$0 = v_i \sin\theta - g\Delta t$$
$$\Delta t = \frac{v_i \sin\theta}{g}$$
$$= \frac{(25 \text{ m/s})(\sin 30.0°)}{9.8 \text{ m/s}^2}$$
$$\Delta t = 1.28 \text{ s (one extra digit carried)}$$
$$\Delta d_{y\,max} = (v_i \sin\theta)\Delta t - \frac{1}{2}g\Delta t^2$$
$$= (25 \text{ m/}\cancel{\text{s}})(\sin 30.0°)(1.28 \cancel{\text{s}}) - \frac{1}{2}(9.8 \text{ m/}\cancel{\text{s}^2})(1.28 \cancel{\text{s}})^2$$
$$\Delta d_{y\,max} = 8.0 \text{ m}$$

Statement: The maximum height of the ball is 8.0 m.

(b) **Given:** $v_i = 25$ m/s; $\Delta d_y = 14$ m; $\theta = 30.0°$

Required: v_f

Analysis: Set $d_i = 0$ as the point at which the golfer strikes the golf ball. Therefore, $\Delta d_y = -14$ m. Use the equation $v_{fy}^2 = v_{iy}^2 - 2g\Delta d_y$ to calculate the final vertical velocity of the ball before it hits the ground. Then, calculate the velocity when the ball lands using $v_f = \sqrt{v_{fx}^2 + v_{fy}^2}$ and the inverse tangent ratio.

Solution:

$$v_{fy}^2 = v_{iy}^2 - 2g\Delta d_y$$
$$= (v_i \sin\theta)^2 - 2g\Delta d_y$$
$$v_{fy} = \pm\sqrt{((25 \text{ m/s})(\sin 30.0°))^2 - 2(9.8 \text{ m/s}^2)(-14 \text{ m})}$$
$$= \pm 20.8 \text{ m/s}$$
$$v_{fy} = -20.8 \text{ m/s (negative because the object is moving down)}$$
$$v_x = v_i \cos\theta$$
$$= 25(\cos 30.0°)$$
$$v_x = 21.7 \text{ m/s}$$
$$v_f = \sqrt{v_x^2 + v_y^2}$$
$$= \sqrt{(21.7 \text{ m/s})^2 + (-20.8 \text{ m/s})^2}$$
$$v_f = 30.1 \text{ m/s}$$
$$\theta = \tan^{-1}\left(\frac{|v_y|}{|v_x|}\right)$$
$$= \tan^{-1}\left(\frac{20.8}{21.7}\right)$$
$$\theta = 44°$$

Statement: The velocity of the ball when it lands is 30.1 m/s [44° below the horizontal].

Practice

1. A marble rolls off a table with a horizontal velocity of 1.93 m/s and onto the floor. The tabletop is 76.5 cm above the floor. Air resistance is negligible. K/U T/I A
   (a) Determine how long the marble is in the air. [ans: 0.40 s]
   (b) Calculate the range of the marble. [ans: 76 cm]
   (c) Calculate the velocity of the marble when it hits the floor. [ans: 4.3 m/s [64° below the horizontal]]
2. A baseball pitcher throws a ball horizontally. The ball falls 83 cm while travelling 18.4 m to home plate. Calculate the initial horizontal speed of the baseball. Air resistance is negligible. K/U T/I A [ans: 45 m/s]
3. In a children's story, a princess trapped in a castle wraps a message around a rock and throws it from the top of the castle. Right next to the castle is a moat. The initial velocity of the rock is 12 m/s [42° above the horizontal]. The rock lands on the other side of the moat, at a level 9.5 m below the initial level. Air resistance is negligible. K/U T/I A
   (a) Calculate the rock's time of flight. [ans: 2.4 s]
   (b) Calculate the width of the moat. [ans: 22 m]
   (c) Determine the rock's velocity on impact with the ground. [ans: 18 m/s [61° below the horizontal]]
4. A friend tosses a baseball out of his second-floor window with an initial velocity of 4.3 m/s [42° below the horizontal]. The ball starts from a height of 3.9 m, and you catch the ball 1.4 m above the ground. K/U T/I A
   (a) Calculate the time the ball is in the air. [ans: 0.48 s]
   (b) Determine your horizontal distance from the window. [ans: 1.5 m]
   (c) Calculate the speed of the ball as you catch it. [ans: 8.2 m/s]

The Range Equation

When you know the initial velocity and the launch angle of a projectile, you can calculate the projectile's range (Δd_x). Now suppose we launch a projectile that lands at the same height it started from, as shown in **Figure 6**. In this case, $\Delta d_y = 0$, and we can use this fact to significantly simplify the equations of motion. We can calculate the range using the equation $\Delta d_x = v_{ix}\Delta t$ if we know the initial velocity and launch angle.

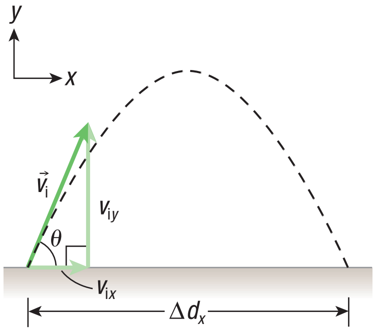


Figure 6 The projectile lands at the same height from which it was launched.

To determine the value of Δt, use the equation for vertical motion and $v_{iy} = v_i \sin\theta$:

$$\Delta d_y = (v_i \sin\theta)\Delta t - \frac{1}{2}g\Delta t^2$$

The final level is the same as the initial level, so $\Delta d_y = 0$. Substituting values in the vertical motion equation gives

$$0 = (v_i \sin\theta)\Delta t - \frac{1}{2}g\Delta t^2$$

$$0 = \Delta t\left(v_i \sin\theta - \frac{1}{2}g\Delta t\right)$$

Therefore, either $\Delta t = 0$ on takeoff or $v_i \sin\theta - \frac{1}{2}g\Delta t = 0$ on landing. Solving the latter equation for Δt gives the following equation:

$$\Delta t = \frac{2v_i \sin\theta}{g}$$

Now, we return to the equation for range, $\Delta d_x = v_i\Delta t$. Substituting Δt and the initial velocity in the x-direction, $v_{ix} = v_i \cos\theta$, gives

$$\Delta d_x = v_{ix}\Delta t$$

$$= v_i \cos\theta\left(\frac{2v_i \sin\theta}{g}\right)$$

$$\Delta d_x = \frac{v_i^2}{g}2\sin\theta\cos\theta$$

Substituting the trigonometry identity $2\sin\theta\cos\theta = \sin 2\theta$ into the above equation, we get the following for the range of a projectile:

$$\Delta d_x = \frac{v_i^2}{g}\sin 2\theta$$

where v_i is the magnitude of the initial velocity of a projectile launched at an angle θ to the horizontal. Note that this equation applies only when $\Delta d_y = 0$, that is, only when the projectile lands at the same height from which it was launched. The largest value of the range is when $\sin 2\theta = 1$ because the sine function has a maximum value of 1. This maximum value occurs when the angle is 90°. Since $2\theta = 90°$, then $\theta = 45°$, so the largest value the range can have occurs when $\theta = 45°$.

UNIT TASK **BOOKMARK**

If your extreme sport uses projectile motion, you can use the ideas in this section to complete the Unit Task on page 146.

All the previous discussion and examples of projectile motion have assumed that air resistance is negligible. This is close to the true situation in cases involving relatively dense objects moving at low speeds, such as a shot used in a shot put competition. However, for many situations you cannot ignore air resistance. When you consider air resistance, the analysis of projectile motion becomes more complex and is beyond the scope of this text.

In Tutorial 2, you will use the kinematics equations to calculate the maximum height and the range for a projectile that lands at the launching height.

Tutorial 2 Solving Projectile Motion Problems

Some projectile motion problems involve an object that starts and ends at the same height and is propelled at an angle above the horizontal. This Tutorial models how to solve projectile motion problems of this type.

Sample Problem 1: Solving Projectile Motion Problems in Which the Object Lands at the Same Height as the Launching Height

Suppose you kick a soccer ball at 28 m/s toward the goal at a launch angle of 21°.

(a) How long does the soccer ball stay in the air?

(b) Determine the distance the soccer ball would need to cover to score a goal (the range).

Solution

(a) **Given:** $v_i = 28$ m/s; $\theta = 21°$

Required: Δt

Analysis: $\Delta t = \dfrac{2v_i \sin\theta}{g}$

Solution:
$$\Delta t = \frac{2v_i \sin\theta}{g} = \frac{2(28 \text{ m/s}) \sin 21°}{9.8 \text{ m/s}^2}$$
$$\Delta t = 2.0 \text{ s}$$

Statement: The soccer ball stays in the air for 2.0 s.

(b) **Given:** $v_i = 28$ m/s; $\theta = 21°$

Required: Δd_x

Analysis: $\Delta d_x = \dfrac{v_i^2}{g}\sin 2\theta$

Solution:
$$\Delta d_x = \frac{v_i^2}{g}\sin 2\theta = \frac{(28 \text{ m/s})^2}{9.8 \text{ m/s}^2}\sin(2(21°))$$
$$\Delta d_x = 54 \text{ m}$$

Statement: The range of the soccer ball is 54 m.

Practice

1. A projectile launcher is set at an angle of 45° above the horizontal and fires an object with a speed of 2.2×10^2 m/s. The object lands at the same height from which it was launched. Air resistance is negligible. Calculate the object's
   (a) time of flight [ans: 32 s]
   (b) horizontal range [ans: 4.9×10^3 m]
   (c) maximum height [ans: 1.2×10^3 m]
   (d) velocity at impact with the ground K/U T/I A [ans: 2.2×10^2 m/s [45° below the horizontal]]
2. A projectile is launched with an initial speed of 14.5 m/s at an angle of 35.0° above the horizontal. The object lands at the same height from which it was launched. Air resistance is negligible. Determine
   (a) the projectile's maximum height [ans: 3.5 m]
   (b) the projectile's horizontal displacement when it hits the ground [ans: 2.0×10^1 m]
   (c) how long the projectile takes to reach its maximum height K/U T/I A [ans: 0.85 s]
3. What happens to each of the following when the initial velocity of a projectile is doubled? Assume the projectile lands at the same height from which it was launched. K/U T/I A
   (a) the time of flight
   (b) the range
   (c) the maximum height

1.5 Review

Summary

- A projectile is an object that moves along a trajectory through the air, with only the force of gravity acting on it.
- An object moving with projectile motion has a constant horizontal velocity and a constant vertical acceleration.
- The time that a projectile moves in the horizontal direction is the same as the time that it moves in the vertical direction.
- When an object lands at the same height from which it was launched, use the range equation to determine the horizontal range: $\Delta d_x = \frac{v_i^2}{g} \sin 2\theta$.

Questions

1. A rock kicked horizontally off a cliff moves 8.3 m horizontally while falling 1.5 m vertically. Calculate the rock's initial speed. K/U T/I A
2. A projectile launcher sends an object with an initial velocity of 1.1×10^3 m/s [45° above the horizontal] into the air. The launch level is at the same level as the landing level. K/U T/I A
   (a) Calculate how long the object is airborne.
   (b) Determine its maximum range.
   (c) Determine the maximum height of the object.
3. In a physics demonstration, a volleyball is tossed from a window at 6.0 m/s [32° below the horizontal], and it lands 3.4 s later. Calculate (a) the height of the window and (b) the velocity of the volleyball at ground level. K/U T/I A
4. A person kicks a soccer ball with an initial velocity directed 53° above the horizontal. The ball lands on a roof 7.2 m high. The wall of the building is 25 m away, and it takes the ball 2.1 s to pass directly over the wall. K/U T/I A
   (a) Calculate the initial velocity of the ball.
   (b) Determine the horizontal range of the ball.
   (c) By what vertical distance does the ball clear the wall of the building?
5. A small asteroid strikes the surface of Mars and causes a rock to fly upward with a velocity of 26 m/s [52° above the horizontal]. The rock rises to a maximum height and then lands on the side of a hill 12 m above its initial position. The acceleration due to gravity on the surface of Mars is 3.7 m/s^2. K/U T/I A
   (a) Calculate the maximum height of the rock.
   (b) Determine the time that the rock is in flight.
   (c) What is the range of the rock?
6. A rock is thrown at an angle of 65° above the horizontal at 16 m/s up a hill that makes an angle of 30° with the horizontal. How far up the hill will the rock go before hitting the ground? K/U T/I A
7. A projectile launcher launches a snowball at 45 m/s from the top of building 1 in **Figure 7**. Does the snowball land on top of building 2? Support your answer with calculations. T/I

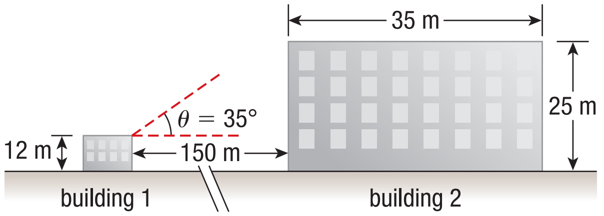


Figure 7

8. In a physics demonstration, a projectile launcher on the floor is aimed directly at a target hanging from the ceiling on the other side of the room (**Figure 8**). When the projectile is launched, the target is released at exactly the same time and the projectile hits the target. Explain why the projectile will always hit the target as long as it reaches the target before they strike the floor. K/U T/I C A

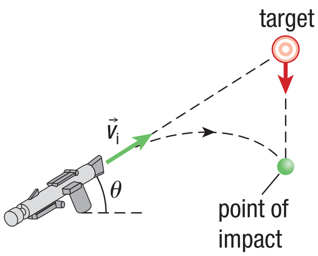


Figure 8

9. A football is thrown from the edge of a cliff from a height of 22 m at a velocity of 18 m/s [39° above the horizontal]. A player at the bottom of the cliff is 12 m away from the base of the cliff and runs at a maximum speed of 6.0 m/s to catch the ball. Is it possible for the player to catch the ball? Support your answer with calculations. K/U T/I A