2.3 Projectile Motion

How would you describe the motion of the Olympic ski jumper shown in **Figure 1** as he begins his ski jump? What path will his motion take as he falls toward the ground? The motion experienced by a ski jumper is identical to that of a ball thrown up in the air at an angle. Both travel through a two-dimensional curved path called a parabola. Any object that moves in response to gravity along a two-dimensional curved trajectory is called a **projectile**. The motion of a projectile under gravity is called **projectile motion**.

Figure 1 An Olympic ski jumper uses his own body as a projectile.

Imagine you and a friend are standing in front of an open window. You each have an identical rubber ball. At the same instant, you throw your rubber ball horizontally out the window while your friend allows her rubber ball to just fall to the ground. Which rubber ball will reach the ground first? The answer may be surprising: both rubber balls will reach the ground at exactly the same time.

To understand this situation, recall how in river crossing problems the boat experiences two velocities that are perpendicular to each other. Since the velocity across the river and the velocity down the river have no effect on each other, these two velocities are independent.

In the example above, both rubber balls experience a vertical acceleration due to gravity. Although your rubber ball is projected horizontally, its horizontal motion does not affect its vertical motion. This is because the projectile's horizontal velocity is perpendicular to its vertical velocity. These two velocities are independent of each other. **Figure 2** shows that the horizontal velocity does not change, while the vertical velocity increases from zero with uniform acceleration. The rubber ball that you throw horizontally will experience the same vertical motion as the rubber ball that falls straight down. As a result, both rubber balls reach the ground at the same time. The amount of time it takes for a projectile to complete its motion is known as its **time of flight**.

projectile an object that moves along a two-dimensional curved trajectory in response to gravity

projectile motion the motion of a projectile under gravity

time of flight the time taken for a projectile to complete its motion

Investigation 2.3.1

Modelling Projectile Motion (p. 86)
In this investigation, you will launch projectiles horizontally from various heights and analyze your results using what you have learned about projectile motion.

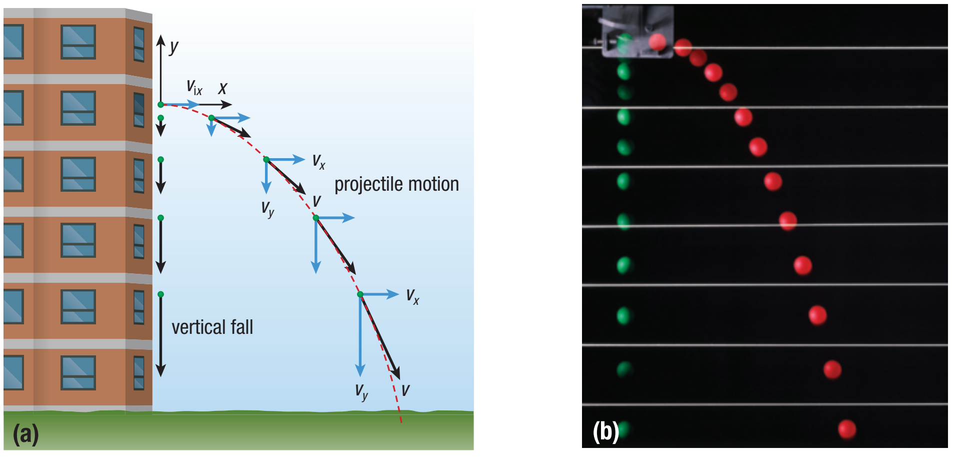

Figure 2 (a) The motion of one rubber ball dropped and the second rubber ball projected horizontally at a constant speed from the same height (b) A time-lapse image of one ball dropped and the other projected horizontally

LEARNING TIP

x*, *y*, and *t
In projectile motion problems, it is helpful to remember that time, *t*, is a scalar variable and is different from either the *x*-variable or the *y*-variable describing the two-dimensional vectors of displacement, velocity, and acceleration.

There is, however, one key difference between a river crossing problem and the projectile motion problem described above. In a river crossing problem both velocities are constant. In a projectile motion problem, while the horizontal velocity is constant, the vertical velocity changes because of the acceleration due to gravity. The rubber ball that you throw is simultaneously undergoing uniform velocity horizontally and uniform acceleration vertically. These two motions are independent of each other, but once again they do share one common factor: time. The time taken for the horizontal motion is exactly the same as the time taken for the vertical motion. This is true since the projectile comes to rest when it hits the ground. The horizontal distance travelled by a projectile (Δd_x) is known as the **range**.

range the horizontal distance travelled by a projectile

Describing Projectile Motion

Projectile motion problems are two-dimensional vector problems. To describe motion in this type of problem in terms of vectors, we will use the convention that velocity vectors pointing upward or to the right have positive values and velocity vectors pointing downward or to the left have negative values (**Figure 3**).

One of the techniques that we will use in solving projectile motion problems is to work with motion in only one direction (horizontal or vertical) at a time. By doing so, we will use information provided about motion in one direction to solve for a time value. This time value can then be used to calculate values for the other direction.

One of the simplest types of projectile motion is when an object is projected horizontally from a known height. This scenario is covered in Tutorial 1. In Tutorial 2, we will consider the case when an object is projected at an angle to the horizontal.

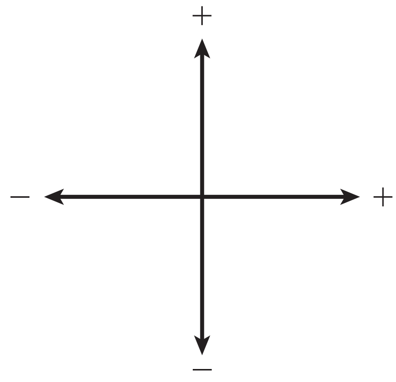

Figure 3 Sign conventions for projectile motion

Tutorial 1 Launching a Projectile Horizontally

Since we will be working with only one motion at any given time, we will not use vector arrows in these problems. Remember, however, that projectile motion problems are vector problems. When a projectile is launched horizontally, it has an initial velocity $\vec{v}_{ix}$ in the horizontal direction. The initial velocity in the vertical direction $\vec{v}_{iy}$ is equal to zero.

Sample Problem 1

A beanbag is thrown from a window 10.0 m above the ground with an initial horizontal velocity of 3.0 m/s.

(a) How long will it take the beanbag to reach the ground? That is, what is its time of flight?

(b) How far will the beanbag travel horizontally? That is, what is its range?

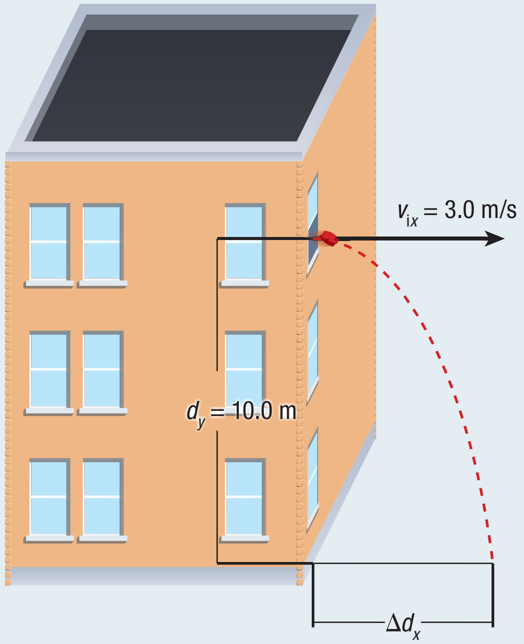

Figure 4 Projectile motion, launching horizontally

Solution

(a) To determine the time of flight of the beanbag, consider its vertical motion.

Given: $\Delta d_y = -10.0$ m; $a_y = -9.8$ m/s^2; $v_{iy} = 0$ m/s

Required: Δt

Analysis: We can use one of the five motion equations from Section 1.5 to solve for the time it takes the beanbag to reach the ground:

$$\Delta d_y = v_{iy}\Delta t + \frac{1}{2}a_y\Delta t^2$$

Solution: In **Figure 4**, notice how the beanbag undergoes motion in the shape of a parabola. Notice also that in the given statement, the vertical displacement is shown as negative. This indicates that the beanbag is falling downward. Similarly, acceleration due to gravity is given as negative. The initial velocity in the vertical direction (v_{iy}) is zero because the beanbag is not thrown upward or downward.

$$\Delta d_y = v_{iy}\Delta t + \frac{1}{2}a_y\Delta t^2$$

$$= 0 + \frac{1}{2}a_y\Delta t^2$$

$$\Delta d_y = \frac{a_y\Delta t^2}{2}$$

$$\Delta t = \sqrt{\frac{2\Delta d_y}{a_y}}$$

$$= \sqrt{\frac{2(-10.0\ \cancel{\text{m}})}{-9.8\frac{\cancel{\text{m}}}{\text{s}^2}}}$$

$$\Delta t = 1.43\ \text{s}$$

Statement: It takes 1.4 s for the beanbag to reach the ground.

Notice that an extra digit has been included in the calculated answer for Δt. This is because the value of Δt will be used in the next calculation, and we wish to minimize rounding error.

(b) To determine the beanbag's horizontal distance or range, we will consider its horizontal motion. We will use the fact that both motions, vertical and horizontal, take the same amount of time.

Given: $\Delta t = 1.43$ s; $v_{ix} = 3.0$ m/s; $a_x = 0$ m/s^2

Notice that the time value is the same as for the vertical motion. The horizontal acceleration is zero, since the beanbag is not experiencing any force in the horizontal direction.

Required: Δd_x

Analysis: $\Delta d_x = v_{ix}\Delta t$

Solution: $\Delta d_x = v_{ix}\Delta t$

$$= (3.0 \text{ m/s})(1.43 \text{ s})$$

$$\Delta d_x = 4.3 \text{ m}$$

Statement: The beanbag travels 4.3 m horizontally.

Practice

1. A hockey puck is launched horizontally from the roof of a 32 m tall building at a velocity of 8.6 m/s. T/I
   (a) What is the hockey puck's time of flight? [ans: 2.6 s]
   (b) What is the hockey puck's range? [ans: 22 m]
2. Suppose the hockey puck in Question 1 has an initial velocity one-half the value given. How does this affect the puck's time of flight and range? K/U

You can increase the range of the beanbag in Tutorial 1 by projecting it partially upward instead of horizontally. In other words, you can give v_{iy} and v_{ix} values other than zero. By doing so, you can increase the time of flight for the projectile. Since the projectile is moving horizontally at a constant velocity, increasing the time of flight can increase the horizontal displacement. This is why competitive swimmers begin their races by diving slightly upward as well as forward. Increasing the launch angle also decreases the horizontal velocity, however. So if you choose too large an angle, you may find that the range of your projectile actually decreases (**Figure 5**).

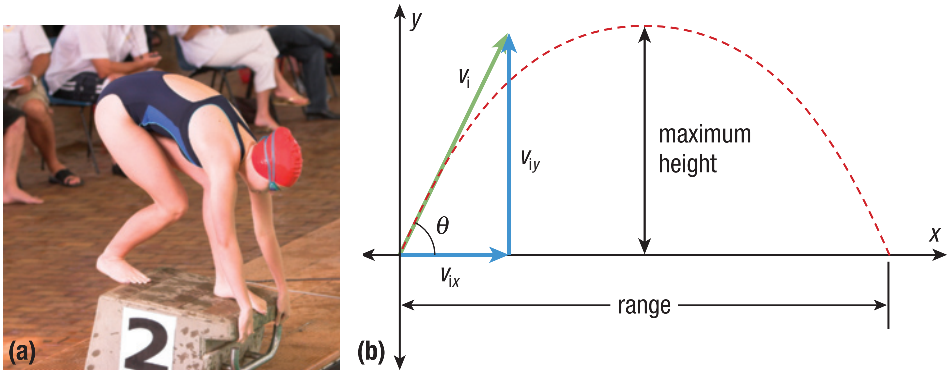

Figure 5 (a) A competitive swimmer (b) Competitive swimmers judge their launch angle carefully to maximize their range before entering the pool.

Tutorial 2 Launching a Projectile at an Angle to the Horizontal

Projectiles that are launched at an angle to the horizontal also undergo parabolic motion. The calculations in this tutorial are similar to those in Tutorial 1. The main difference is that the projectile has an initial velocity in the vertical direction (v_{iy}). This is because the object is launched at an angle rather than horizontally.

Sample Problem 1

A soccer player running on a level playing field kicks a soccer ball with a velocity of 9.4 m/s at an angle of 40° above the horizontal. Determine the soccer ball's

(a) time of flight
(b) range
(c) maximum height

Figure 6 shows the soccer ball being kicked from ground level, undergoing parabolic motion, and eventually landing back on the ground. Notice that for this situation the total vertical displacement of the ball (Δd_y) is zero.

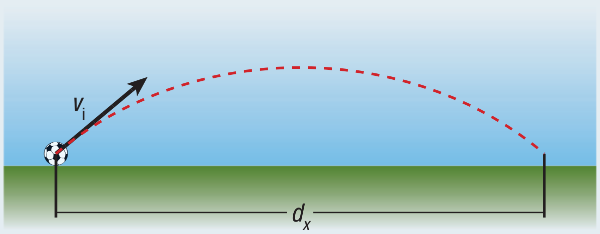

Figure 6 Motion of the soccer ball

Figure 7 shows the initial velocity of the ball broken down into its vertical and horizontal components. We can determine the magnitude of these two components by recalling that $v_{ix} = v_i \cos 40°$ and $v_{iy} = v_i \sin 40°$.

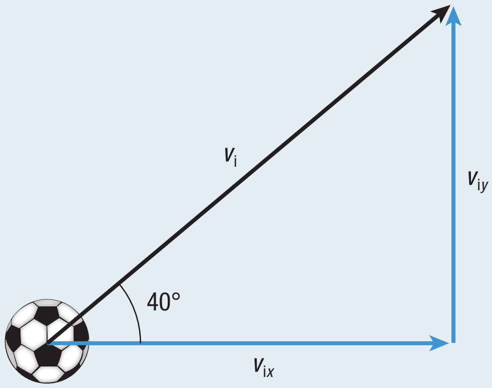

Figure 7 Components of the initial velocity

(a) Consider the soccer ball's vertical motion:

Given: $\Delta d_y = 0$ m; $a_y = -9.8$ m/s^2; $v_i = 9.4$ m/s

Required: time of flight

Analysis: $\Delta d_y = v_{iy}\Delta t + \frac{1}{2}a_y\Delta t^2$

Solution: $\Delta d_y = v_{iy}\Delta t + \frac{1}{2}a_y\Delta t^2$

$$0 = v_i(\sin 40°)\Delta t + \frac{1}{2}a_y\Delta t^2$$

$$0 = v_i(\sin 40°) + \frac{1}{2}a_y\Delta t, \ \Delta t \neq 0$$

(dividing both sides by Δt)

$$\frac{a_y\Delta t}{2} = -v_i(\sin 40°)$$

$$\Delta t = \frac{-2v_i(\sin 40°)}{a_y}$$

$$= \frac{-2(9.4 \text{ m/s})(\sin 40°)}{-9.8 \text{ m/s}^2}$$

$$\Delta t = 1.233 \text{ s}$$

Statement: The soccer ball's time of flight is 1.2 s.

(b) Consider the horizontal motion:

Given: $\Delta t = 1.233$ s; $v_i = 9.4$ m/s; $a_x = 0$ m/s^2

Required: Δd_x

Analysis: Since the ball is travelling at a constant velocity horizontally, we can use the defining equation for average velocity to calculate the range.

$$\Delta d_x = v_{ix}\Delta t$$

Solution: $\Delta d_x = v_{ix}\Delta t$

$$= v_i(\cos 40°)\Delta t$$

$$= \left(9.4 \frac{\text{m}}{\cancel{\text{s}}}\right)(\cos 40°)(1.233 \cancel{\text{s}})$$

$$\Delta d_x = 8.9 \text{ m}$$

Statement: The soccer ball's range is 8.9 m.

(c) We can analyze the vertical motion to determine the maximum height of the ball. If we only consider the motion of the ball on its way up to its maximum height, we know its initial velocity and its acceleration. We also know that at its maximum height, the ball will come to rest in the vertical or y-direction. As a result, its final vertical velocity (v_{fy}) (considering the motion only as far as the maximum height reached) is zero.

Consider the vertical motion:

Given: $a_y = -9.8$ m/s^2; $v_i = 9.4$ m/s; $v_{fy} = 0$ m/s

Required: Δd_y

Since the ball is undergoing uniform vertical acceleration, we can use one of the five key kinematics equations to solve for the vertical displacement at maximum height.

Analysis: $v_{fy}^2 = v_{iy}^2 + 2a_y\Delta d_y$

Solution: $v_{fy}^2 = v_{iy}^2 + 2a_y\Delta d_y$

$$0 = v_{iy}^2 + 2a_y\Delta d_y$$

$$\Delta d_y = \frac{-v_{iy}^2}{2a_y}$$

$$= \frac{-[(9.4 \text{ m/s})(\sin 40°)]^2}{2(-9.8 \text{ m/s}^2)}$$

$$\Delta d_y = 1.9 \text{ m}$$

Statement: The soccer ball's maximum height is 1.9 m.

The most complex type of projectile motion problem combines the previous two problem types. In this situation the projectile is launched at an angle from a height above the ground and lands at another height. This is the scenario that we will consider in the next Sample Problem.

Sample Problem 2

A golfer is trying to improve the range of her shot. To do so she drives a golf ball from the top of a steep cliff, 30.0 m above the ground where the ball will land. If the ball has an initial velocity of 25 m/s and is launched at an angle of 50° above the horizontal, determine the ball's time of flight, its range, and its final velocity just before it hits the ground. **Figure 8** shows the motion of the golf ball.

For this solution we will combine the horizontal and vertical given statements.

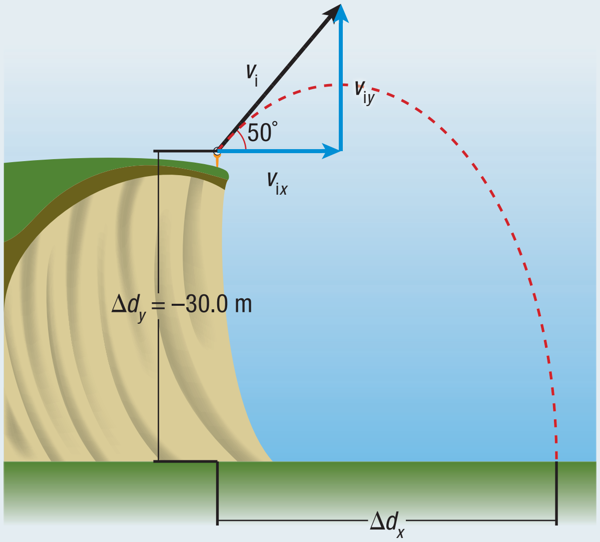

Figure 8 Motion of the golf ball

Given: $\Delta d_y = -30.0$ m; $a_y = -9.8$ m/s^2; $v_i = 25$ m/s; $a_x = 0$ m/s^2; $\theta = 50°$

To determine the horizontal range, we first need to determine the time of flight. Consider the vertical motion.

Required: Δt

Analysis: $\Delta d_y = v_{iy}\Delta t + \frac{1}{2}a_y\Delta t^2$

Notice that this is a quadratic equation for time. Previously, whenever we needed to solve this equation for time, one of the variables in this equation was equal to zero. This allowed us to solve for Δt without having to solve a quadratic equation. In this more complicated case it is necessary to solve the quadratic equation. We must therefore expand this equation and use the quadratic formula. To simplify the calculation, we will ignore the units until the end.

Solution: $\Delta d_y = v_{iy}\Delta t + \frac{1}{2}a_y\Delta t^2$

$$= v_i(\sin\theta)\Delta t + \frac{1}{2}a_y\Delta t^2$$

$$-30.0 = 25\sin 50°\Delta t + \frac{1}{2}(-9.8)\Delta t^2$$

$$0 = -4.9\Delta t^2 + 19.2\Delta t + 30.0$$

$$\Delta t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Delta t = \frac{-19.2 \pm \sqrt{(19.2)^2 - 4(-4.9)(30.0)}}{2(-4.9)}$$

$$= \frac{-19.2 \pm 30.930}{-9.8}$$

$\Delta t = -1.197$ s or $\Delta t = 5.115$ s (two extra digits carried)

We will take the positive root because negative time has no meaning in this context.

Statement: The golf ball's time of flight is 5.1 s.

To determine the range, consider the horizontal motion:

Required: Δd_x

Analysis: $\Delta d_x = v_{ix}\Delta t$

Solution: $\Delta d_x = v_i(\cos 50°)\Delta t$

$$= (25\text{ m/s})(\cos 50°)(5.11\text{ s})$$

$$\Delta d_x = 82\text{ m}$$

Statement: The range of the golf ball is 82 m.

To determine the final velocity of the ball just before it hits the ground, consider **Figure 9**. This figure shows the final velocity of the ball as well as its horizontal and vertical components. Since the golf ball was travelling at a constant horizontal velocity, we know that the final horizontal velocity (v_{fx}) equals the initial horizontal velocity (v_{ix}). In the vertical direction, however, the initial and final velocities are not the same since the golf ball is accelerating vertically.

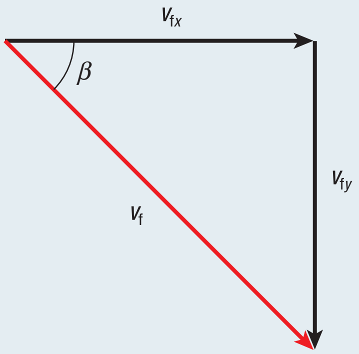

Figure 9 Determining the final velocity

Consider the horizontal motion:

$$v_{fx} = v_{ix}$$
$$= v_i\cos 50°$$
$$= (25\text{ m/s})\cos 50°$$
$$v_{fx} = 16.1\text{ m/s}$$

Consider the vertical motion:

$$v_{fy}^2 = v_{iy}^2 + 2a_y\Delta d_y$$
$$v_{fy} = \sqrt{v_{iy}^2 + 2a_y\Delta d_y}$$
$$= \sqrt{(v_i\sin 50°)^2 + 2a_y\Delta d_y}$$
$$= \sqrt{[(25\text{ m/s})(\sin 50°)]^2 + 2(-9.8\text{ m/s}^2)(-30.0\text{ m})}$$
$$v_{fy} = 30.9\text{ m/s}$$

We can then determine the final velocity by using the Pythagorean theorem and the inverse tangent function:

$$v_f^2 = v_{fx}^2 + v_{fy}^2$$

$$v_f = \sqrt{v_{fx}^2 + v_{fy}^2}$$

$$= \sqrt{(16.1 \text{ m/s})^2 + (30.9 \text{ m/s})^2}$$

$$v_f = 35 \text{ m/s}$$

$$\tan \beta = \frac{v_{fy}}{v_{fx}}$$

$$\tan \beta = \frac{30.9 \text{ m/s}}{16.1 \text{ m/s}}$$

$$\beta = 62°$$

$$\vec{v}_f = 35 \text{ m/s [right 62° down]}$$

Statement: The golf ball's final velocity is 35 m/s [right 62° down].

Practice

1. A superhero launches himself from the top of a building with a velocity of 7.3 m/s at an angle of 25° above the horizontal. If the building is 17 m high, how far will he travel horizontally before reaching the ground? What is his final velocity? [ans: 15 m; 20 m/s [right 70° down]] T/I
2. Two identical baseballs are initially at the same height. One baseball is thrown at an angle of 40° above the horizontal. The other baseball is released at the same instant and is allowed to fall straight down. Compare the amount of time it takes for the two baseballs to reach the ground. Explain your answer. K/U C

2.3 Summary

- Projectile motion consists of independent horizontal and vertical motions.
- The horizontal and vertical motions of a projectile take the same amount of time.
- Projectiles move horizontally at a constant velocity. Projectiles undergo uniform acceleration in the vertical direction. This acceleration is due to gravity.
- Objects can be projected horizontally or at an angle to the horizontal. Projectile motion can begin and end at the same or at different heights.
- The five key equations of motion can be used to solve projectile motion problems. The time of flight, range, and maximum height can all be determined given the initial velocity and the vertical displacement of the projectile.

UNIT TASK **BOOKMARK**

You can use what you have learned about angle of projection and range to calibrate your launcher in the Unit Task on page 96.

2.3 Questions

1. What do the horizontal and vertical motions of a projectile have in common? K/U
2. A tennis ball thrown horizontally from the top of a water tower lands 20.0 m from the base of the tower. If the tennis ball is initially thrown at a velocity of 10.0 m/s, how high is the water tower? How long does it take the tennis ball to reach the ground? T/I
3. At what angle should you launch a projectile from the ground so that it has the
   (a) greatest time of flight?
   (b) greatest range, assuming no air resistance? (Hint: Use your findings from Investigation 2.3.1) K/U
4. A field hockey ball is launched from the ground at an angle to the horizontal. What are the ball's horizontal and vertical accelerations
   (a) at its maximum height?
   (b) halfway up to its maximum height?
   (c) halfway down to the ground? K/U
5. An archer shoots at a target 60 m away. If she shoots at a velocity of 55 m/s [right] from a height of 1.5 m, does the arrow reach the target before striking the ground? What should the archer do to get her next shot on target? T/I C
6. An acrobat is launched from a cannon at an angle of 60° above the horizontal. The acrobat is caught by a safety net mounted horizontally at the height from which he was initially launched. Suppose the acrobat is launched at a speed of 26 m/s. T/I
   (a) How long does it take before he reaches his maximum height?
   (b) How long does it take in total for him to reach a point halfway back down to the ground?
7. A championship golfer uses a nine iron to chip a shot right into the cup. If the golf ball is launched at a velocity of 20 m/s at an angle of 45° above the horizontal, how far away was the golfer from the hole when he hit the ball? What maximum height did the ball reach? T/I
8. As part of a physics investigation, a student launches a beanbag out of an open window with a velocity of 4.5 m/s at an angle of 25° above the horizontal. If the window is 12 m above the ground, how far away from the building must the student's friend stand to catch the beanbag at ground level? T/I