1.5

Five Key Equations for Motion with Uniform Acceleration

Graphical analysis is an important tool for physicists to use to solve problems. Sometimes, however, we have enough information to allow us to solve problems algebraically. Algebraic methods tend to be quicker and more convenient than graphical analysis. For example, if you want to determine how far a passing vehicle would travel in a given amount of time, you could perform an experiment using a motion sensor. You would collect position–time data with the motion sensor and then plot the data on a graph. From the graph, you could then measure how far the vehicle has gone in a given amount of time. However, if you were in the vehicle, you would simply use the vehicle's speedometer to determine the speed of the vehicle. Knowing the speed of your vehicle, you could easily determine how far it would travel in a given time interval using the equation $v_{av} = \frac{\Delta d}{\Delta t}$. As you can see, the best way to solve a problem is usually determined by the information that is available to you.

To be able to solve problems related to motion with uniform acceleration, in which the velocity may change but the acceleration is constant, we need to derive algebraic equations that describe this type of motion. We will start with equations that we have already used in previous sections.

A Displacement Equation for Uniformly Accelerated Motion

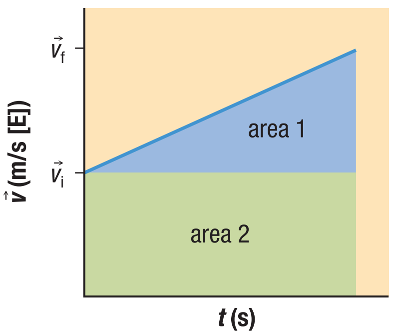

Figure 1 A velocity–time graph for an object undergoing uniform acceleration

The velocity–time graph in **Figure 1** shows a straight line with a non-zero intercept. This graph is a non-horizontal straight line, showing that the object is undergoing uniform, or constant, acceleration. In other words, the velocity is increasing at a uniform, or constant, rate. We know that to determine the displacement of this object from the velocity–time graph, we must determine the area under the line. For the graph in Figure 1, we must determine the area of a rectangle and a triangle:

$$\begin{aligned}
\Delta\vec{d} &= A_{\text{triangle}} + A_{\text{rectangle}} \\
&= \frac{1}{2}bh + lw \\
&= \frac{1}{2}\Delta t(\vec{v}_f - \vec{v}_i) + \Delta t\vec{v}_i \\
&= \frac{1}{2}\vec{v}_f\Delta t - \frac{1}{2}\vec{v}_i\Delta t + \vec{v}_i\Delta t \\
&= \frac{1}{2}\vec{v}_f\Delta t + \frac{1}{2}\vec{v}_i\Delta t
\end{aligned}$$

$$\Delta\vec{d} = \left(\frac{\vec{v}_f + \vec{v}_i}{2}\right)\Delta t \text{ (Equation 1)}$$

LEARNING TIP

Interpreting Areas Under a Motion Graph

Notice that the rectangular area (green) in Figure 1 represents the displacement the object would have undergone had it continued at constant velocity $\vec{v}_i$. The triangular area (blue) represents the extra displacement the object experienced due to its acceleration.

We can use Equation 1 to determine the displacement of an object that is undergoing uniform acceleration. Equation 1 is very similar to an equation that we previously developed from the defining equation for average velocity: $\Delta\vec{d} = \vec{v}_{av}\Delta t$. We can relate the average velocity to the initial and final velocities by the equation

$$\vec{v}_{av} = \left(\frac{\vec{v}_f + \vec{v}_i}{2}\right)$$

Then substituting $\vec{v}_{av}$ in place of $\left(\frac{\vec{v}_f + \vec{v}_i}{2}\right)$ in Equation 1 gives $\Delta\vec{d} = \vec{v}_{av}\Delta t$ directly.

As you will see, Equation 1 can help us to solve many motion problems. However, in some situations we will not know the initial velocity, the final velocity, or the time interval for a given scenario. We could use the defining equation for acceleration in a two-step process, but this tends to be difficult. To simplify things, we can derive a number of other motion equations that will allow us to solve problems in one step.

Additional Motion Equations

Consider the defining equation for acceleration: $\vec{a}_{av} = \dfrac{\vec{v}_f - \vec{v}_1}{\Delta t}$

If we rearrange this equation to solve for final velocity ($\vec{v}_f$), we get Equation 2:

$$\vec{v}_f = \vec{v}_i + \vec{a}_{av}\Delta t \text{ (Equation 2)}$$

You may use Equation 2 in problems that do not directly involve displacement.

If we substitute the expression $v_i + \vec{a}_{av}\Delta t$ from Equation 2 into Equation 1, we get

$$\vec{v}_f = \vec{v}_i + \vec{a}_{av}\Delta t \text{ (Equation 2)}$$

$$\begin{aligned}\Delta\vec{d} &= \left(\frac{\vec{v}_f + \vec{v}_i}{2}\right)\Delta t \text{ (Equation 1)}\\ &= \frac{1}{2}(\vec{v}_i + \vec{a}_{av}\Delta t + \vec{v}_i)\Delta t\\ &= \frac{1}{2}(2\vec{v}_i + \vec{a}_{av}\Delta t)\Delta t\end{aligned}$$

$$\Delta\vec{d} = \vec{v}_i\Delta t + \frac{1}{2}\vec{a}_{av}\Delta t^2 \text{ (Equation 3)}$$

This is Equation 3, which allows you to determine the displacement of an object moving with uniform acceleration given a value for acceleration rather than a final velocity.

The Five Key Equations of Accelerated Motion

Table 1 shows the five key equations of accelerated motion. You should be able to solve any kinematics question by correctly choosing one of these five equations. You have seen how the first three are developed. We will leave the others to be developed as an exercise.

Table 1 The Five Key Equations of Accelerated Motion

	Equation	Variables found in equation	Variables not in equation
Equation 1	$\Delta\vec{d} = \left(\dfrac{\vec{v}_f + \vec{v}_i}{2}\right)\Delta t$	$\Delta\vec{d}$, Δt, $\vec{v}_f$, $\vec{v}_i$	$\vec{a}_{av}$
Equation 2	$\vec{v}_f = \vec{v}_i + \vec{a}_{av}\Delta t$	$\vec{a}_{av}$, Δt, $\vec{v}_f$, $\vec{v}_i$	$\Delta\vec{d}$
Equation 3	$\Delta\vec{d} = \vec{v}_i\Delta t + \dfrac{1}{2}\vec{a}_{av}\Delta t^2$	$\Delta\vec{d}$, $\vec{a}_{av}$, Δt, $\vec{v}_i$	$\vec{v}_f$
Equation 4	$v_f^2 = v_i^2 + 2a_{av}\Delta d$	Δd, a_{av}, v_f, v_i	Δt
Equation 5	$\Delta\vec{d} = \vec{v}_f\Delta t - \dfrac{1}{2}\vec{a}_{av}\Delta t^2$	$\Delta\vec{d}$, $\vec{a}_{av}$, Δt, $\vec{v}_f$	$\vec{v}_i$

Tutorial 1 Using the Five Key Equations of Accelerated Motion

The following Sample Problems will demonstrate how to choose equations and solve problems involving the five key motion equations.

Sample Problem 1

A sports car approaches a highway on-ramp at a velocity of 20.0 m/s [E]. If the car accelerates at a rate of 3.2 m/s^2 [E] for 5.0 s, what is the displacement of the car?

Given: $\vec{v}_i = 20.0 \text{ m/s [E]}$; $\vec{a}_{av} = 3.2 \text{ m/s}^2 \text{[E]}$; $\Delta t = 5.0 \text{ s}$

Required: $\Delta \vec{d}$

Analysis: Our first task is to determine which of the five equations of accelerated motion to use. Usually, you can solve a problem using only one of the five equations. We simply identify which equation contains all the variables for which we have given values and the unknown variable that we are asked to calculate. In Table 1, we see that Equation 3 has all the given variables and will allow us to solve for the unknown variable.

$$\Delta \vec{d} = \vec{v}_i \Delta t + \frac{1}{2}\vec{a}_{av}\Delta t^2$$

Solution: $\Delta \vec{d} = \vec{v}_i \Delta t + \frac{1}{2}\vec{a}_{av}\Delta t^2$

$$= \left(20.0\,\frac{\text{m}}{\cancel{\text{s}}}\text{ [E]}\right)(5.0\,\cancel{\text{s}}) + \frac{1}{2}\left(3.2\,\frac{\text{m}}{\cancel{\text{s}^2}}\text{ [E]}\right)(5.0\,\cancel{\text{s}})^2$$

$$= 100 \text{ m [E]} + 40 \text{ m [E]}$$

$$\Delta \vec{d} = 1.4 \times 10^2 \text{ m [E]}$$

Statement: During the 5.0 s time interval, the car is displaced 1.4 × 10^2 m [E].

Sample Problem 2

A sailboat accelerates uniformly from 6.0 m/s [N] to 8.0 m/s [N] at a rate of 0.50 m/s^2 [N]. What distance does the boat travel?

Given: $v_i = 6.0 \text{ m/s}$; $v_f = 8.0 \text{ m/s}$; $a_{av} = 0.50 \text{ m/s}^2$

Required: Δd

Analysis: In Table 1, we see that Equation 4 will allow us to solve for the unknown variable. First, we rearrange the equation to solve for Δd:

$$v_f^2 = v_i^2 + 2a_{av}\Delta d$$

$$v_f^2 - v_i^2 = 2a_{av}\Delta d$$

$$\Delta d = \frac{v_f^2 - v_i^2}{2a_{av}}$$

Solution: $\Delta d = \dfrac{v_f^2 - v_i^2}{2a_{av}}$

$$= \frac{\left(8.0\,\frac{\text{m}}{\text{s}}\right)^2 - \left(6.0\,\frac{\text{m}}{\text{s}}\right)^2}{2\left(0.50\,\frac{\text{m}}{\text{s}^2}\right)}$$

$$= \frac{(64 - 36)\frac{\text{m}^2}{\cancel{\text{s}^2}}}{1\,\frac{\cancel{\text{m}}}{\cancel{\text{s}^2}}}$$

$$\Delta d = 28 \text{ m}$$

Statement: The boat travels a distance of 28 m.

Sample Problem 3

A dart is thrown at a target that is supported by a wooden backstop. It strikes the backstop with an initial velocity of 350 m/s [E]. The dart comes to rest in 0.0050 s.

(a) What is the acceleration of the dart?

(b) How far does the dart penetrate into the backstop?

For (a):

Given: $\vec{v}_i = 350 \text{ m/s [E]}$; $\vec{v}_f = 0 \text{ m/s}$; $\Delta t = 0.0050 \text{ s}$

Required: $\vec{a}_{av}$

Analysis: We may use the defining equation for acceleration.

$$\vec{a}_{av} = \frac{\vec{v}_f - \vec{v}_i}{\Delta t}$$

Solution: $\vec{a}_{av} = \dfrac{\vec{v}_f - \vec{v}_i}{\Delta t}$

$$= \frac{0\,\frac{\text{m}}{\text{s}} - 350\,\frac{\text{m}}{\text{s}}\text{ [E]}}{0.0050 \text{ s}}$$

$$= -70\,000 \text{ m/s}^2 \text{ [E]}$$

$$\vec{a}_{av} = 7.0 \times 10^4 \text{ m/s}^2 \text{ [W]}$$

Notice that the acceleration is in the opposite direction to the initial motion. This must be true in order for the velocity of the dart to decrease to zero as it comes to rest. If the acceleration were in the same direction as the initial velocity, the final velocity would be greater than the initial velocity.

Statement: The acceleration of the dart is 7.0 × 10^4 m/s^2 [W].

To solve (b), we have sufficient information to solve the problem using any equation with displacement in it. Generally speaking, in a two-part problem like this, it is a good idea to try to find an equation that uses only given information. Then, if we have made an error in calculating the first part (acceleration), our next calculation would be unaffected by the error. Therefore, we will use Equation 1 to solve (b), since it can be solved using only given information.

Given: $\vec{v}_i = 350\text{ m/s [E]}$; $\vec{v}_f = 0\text{ m/s}$; $\Delta t = 0.0050\text{ s}$

Required: $\Delta\vec{d}$

Analysis: $\Delta\vec{d} = \left(\dfrac{\vec{v}_f + \vec{v}_i}{2}\right)\Delta t$

Solution: $\Delta\vec{d} = \left(\dfrac{\vec{v}_f + \vec{v}_i}{2}\right)\Delta t$

$$= \left(\frac{0\,\dfrac{\text{m}}{\cancel{\text{s}}} + 350\,\dfrac{\text{m}}{\cancel{\text{s}}}\,[\text{E}]}{2}\right)(0.0050\,\cancel{\text{s}})$$

$$\Delta\vec{d} = 0.88\text{ m [E]}$$

Statement: The displacement of the dart into the backstop is 0.88 m [E].

Practice

1. A football player initially at rest accelerates uniformly as she runs down the field, travelling 17 m [E] in 3.8 s. What is her final velocity? T/I [ans: 8.9 m/s [E]]
2. A child on a toboggan sits at rest on the top of a tobogganing hill. If the child travels 70.0 m [downhill] in 5.3 s while accelerating uniformly, what acceleration does the child experience? T/I [ans: 5.0 m/s^2 [downhill]]

1.5 Summary

- The five key equations of accelerated motion, listed in Table 1 on page 37, apply to motion with uniform (constant) acceleration. They involve the variables for displacement, initial velocity, final velocity, acceleration, and time interval.
- When solving uniform acceleration problems, choose which equation(s) to use based on the given and required variables of the problem.

1.5 Questions

1. A car accelerates from rest at a rate of 2.0 m/s^2 [N]. What is the displacement of the car at $t = 15$ s? T/I
2. An astronaut is piloting her spacecraft toward the International Space Station. To stop the spacecraft, she fires the retro-rockets, which cause the spacecraft to slow down from 20.0 m/s [E] to 0.0 m/s in 12 s. T/I
   (a) What is the acceleration of the spacecraft?
   (b) What is the displacement of the spacecraft when it comes to rest?
3. A helicopter travelling at a velocity of 15 m/s [W] accelerates uniformly at a rate of 7.0 m/s^2 [E] for 4.0 s. What is the helicopter's final velocity? T/I
4. Two go-carts, A and B, race each other around a 1.0 km track. Go-cart A travels at a constant speed of 20.0 m/s. Go-cart B accelerates uniformly from rest at a rate of 0.333 m/s^2. Which go-cart wins the race and by how much time? T/I
5. A boat increases its speed from 5.0 m/s to 7.5 m/s over a distance of 50.0 m. What is the boat's acceleration? T/I
6. Within 4.0 s of liftoff, a spacecraft that is uniformly accelerating straight upward from rest reaches an altitude of 4.50×10^2 m [up]. T/I
   (a) What is the spacecraft's acceleration?
   (b) At what velocity is the spacecraft travelling when it reaches this altitude?
7. Derive Equation 4 and Equation 5 in Table 1 on page 37 by substituting other expressions. T/I